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Partial Fractions and Arcsin Branch Points

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September 29, 2015

1 PARTIAL FRACTIONS

Direct Computation: A direct computation of the partial fraction coefficients is possible instead of using the lengthy algebraic approach that most of you are familiar with. For instance, assume that you have a fraction $P(x)/Q(x)$, where $Q(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$. Assume that $degP \le degQ$. If that is not the case long division of $P(x)$ by $Q(x)$ needs to be performed first and the formula described below can be used on the remainder from the long division. Now the expansion of $P(x)/Q(x)$ as a sum of partial fractions is done by the following formula:

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{P(\alpha_i)}{Q'(\alpha_i)} \frac{1}{(x-\alpha_i)} \tag{1}$$

where Q' is the derivative of $Q(x)$. You don't have to prove this expression.

2 ARCSIN BRANCH POINTS

Assume the analytic extension of the real function $\arcsin(x)$ to be $\arcsin(z)$ where $z = re^{i\theta}$. What would be the singularities of the function $f(z) = \arcsin(z)$? To answer this question you can try to do the following few exercises.

- **Branch Points of Log:** We will start by looking at the singularities of $\ln(z)$. Express z as $z = re^{i\theta}$ and convince yourself that $\ln(z)$ has a branch point at $z = 0$ and that there is an infinite amount of Riemann sheets for that branch point. This is called a logarithmic branch point (vs. algebraic branch points which have finite number of Riemann sheets.) Now it is also relatively easy to confirm from the same epression that the function has a branch point at infinity. If you can't see that, try substituting $z = \frac{1}{z'}$ and let $z = re^{i\theta'}$ with r' approaching 0.

- **Expressing Arcsin as a function of Log:** We will have to use some clever algebra to express the arcsin(z) in a different form. Use $\sin(\zeta) = z$, where $\zeta = \arcsin(z)$. Now using Euler's formula:

$$\sin(\zeta) = \frac{e^{i\zeta} + e^{-}i\zeta}{2i} \tag{2}$$

  and solve for ζ. You should derive the following expression:

$$\arcsin(z) = -i\ln(iz + (1 - z^2)^{\frac{1}{2}}) \tag{3}$$

  Note that you will have to solve a quadratic equation for ζ. The two solutions of that quadratic equation are hidden behind the branch points at $z = \pm 1$ as it can be seen from equation 3.

- **Branch Points of Arcsin:** Now we are ready to find the branch points of arcsin(z). As it can be seen from equation 3 there will be algorithmic branch points at $z = \pm 1$ because of the fractional $\frac{1}{2}$ exponent. From the previous section we found out that the natural logarithm has a branch point when the argument is zero. Convince yourself that the argument of the logarithm in equation 3 can not be zero for any value of z. Finally, you can show that there is a branch point at infinity by doing similar subtitution like the one described in the previous section: substitute $z = \frac{1}{z'}$ and let $z = re^{i\theta'}$ with r' approaching 0.

- **Conclusion:** The function arcsin(z) has branch points at $z = \pm 1$ (with two Riemann sheets each) and a logarithmic branch point at infinity.