Physics 408 Final Exam Due: May 2, 2016

1. A hydrogen atom is subjected to the additional potential

$$V = A(\mathbf{L}\cdot\mathbf{S})^2$$

where A is a constant and $\mathbf{L}$ and $\mathbf{S}$ are the orbital and spin angular momenta.

a) Give the exact expression for the total energy of the bound states of the system.

b) Give the explicit results for the $n = 1$ and $n = 2$ levels of hydrogen. State the degeneracies of each distinct level.

c) Compute the trace of V for the $n = 1$ and $n = 2$ levels.

d) Show that the results of b) agree with the traces computed in c).

2. Consider the system with Hamiltonian

$$H = AL^2 + B(L_xL_y + L_yL_x)$$

where $B/A \ll 1$.

a) Show that the ground state energy can be computed exactly. What is it?

b) What are the $\ell = 1$ energy levels to lowest nonvanishing order in B?

c) Show that the sum of the eigenvalues must be independent of B for *all* ℓ values.

d) Add the term

$$iC\epsilon_{rst}L_rL_sL_t$$

to H. How does the result of b) change with this replacement?

3. A hydrogen atom is subjected to the potential

$$V = \lambda xyz$$

a) Show that to order λ the $n = 1$ and $n = 2$ levels are unaffected.

b) For $n = 3$ write out the matrix of V. That is, indicate all zero elements of the matrix. Denote nonvanishing elements by constants $a, b, c, \ldots$.

c) Determine all the $n = 3$ energy levels in terms of $a, b, c, \ldots$.

d) Show that the matrix of V can in principle be expressed in terms of a single parameter (basically, a radial integral).

4. Determine the differential cross section for scattering of mass m particles by the square well potential

$$V = \begin{cases} V_0 & r < R \\ 0 & r > R \end{cases}$$

to lowest nonvanishing order in V_0.

1 a) $L\cdot S = \frac{1}{2}(J^2 - L^2 - S^2)$

$$= \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - 3/4\right]$$

$(j = l \pm \tfrac{1}{2},$ except for $l=0)$

$$= \frac{\hbar^2}{2}\left[(l\pm\tfrac{1}{2})(l\pm\tfrac{1}{2}+1) - l(l+1) - 3/4\right]$$

$$= \frac{\hbar^2}{2}\left[\pm l \pm \tfrac{1}{2} - \tfrac{1}{2}\right] = \frac{\hbar^2}{2}(l, -l-1)$$

$$E = -\frac{me^4}{2\hbar^2 n^2} + A\frac{\hbar^4}{4}\begin{cases} l^2 \\ (l+1)^2 \end{cases} \qquad 0, \text{ if } l=0$$

b) $\underline{n=1}$ $\quad E = -\frac{me^4}{2\hbar^2}$ $\quad$ degen $=2$

$\underline{n=2}$

$$E = -\frac{me^4}{8\hbar^2} + A\hbar^4\frac{1}{4}\begin{pmatrix} 0 & l=0 \ (\text{degen}=2) \\ 1 & j=3/2 \ (4) \\ 4 & l=1, j=\tfrac{1}{2} \ (2) \end{pmatrix}$$

c) $\operatorname{Tr} V = 0 \qquad n=1$

$$\operatorname{Tr} V = A(\operatorname{Tr} L_i L_j)\underbrace{(\operatorname{Tr} S_i S_j)}_{2\hbar^2 \delta_{ij}\frac{1}{4}}$$

$$= A\hbar^2 \tfrac{1}{2} \operatorname{Tr} L^2 = \frac{A\hbar^4}{2} 2\cdot 3 = 3A\hbar^4$$

d) $\operatorname{Tr} V = 0 \qquad n=1$

$$\operatorname{Tr} V = \frac{A\hbar^4}{4}\{2\times 0 + 4\times 1 + 2\times 4\} \qquad n=2$$

$$= 3A\hbar^4$$

which checks

2 a) for $l=0$

$$H|0\rangle = 0$$

so E of ground state is zero

b) $L_x = \frac{L_+ + L_-}{2}$, $L_y = \frac{L_+ - L_-}{2i}$

$$H = AL^2 + B\left[\frac{(L_+ + L_-)(L_+ - L_-)}{4i} + \frac{(L_+ - L_-)(L_+ + L_-)}{4i}\right]$$

$$= AL^2 + \frac{B}{4i}\,2\left(L_+^2 - L_-^2\right)$$

matrix of H

$$\hbar^2 \begin{pmatrix} 2A & 0 & -iB \\ 0 & 2A & 0 \\ iB & 0 & 2A \end{pmatrix}$$

so $(2A\hbar^2 - E)^3 - \hbar^2 B^2 (2A\hbar^2 - E) = 0$

so $E = 2A\hbar^2,\ 2A\hbar^2 \pm B\hbar^2$

c) $H - AL^2 = -\frac{iB}{2}\left(L_+^2 - L_-^2\right)$

in l, m representation $\mathrm{Tr}\, L_\pm^2 = 0$

so $\mathrm{Tr}\, H = A\hbar^2\, l(l+1)(2l+1)$

independent of B

d) $iC\epsilon_{rst} L_r L_s L_t = \frac{i}{2} C \epsilon_{rst} L_r [L_s, L_t]$

$= i\,\frac{i\hbar}{2} C\, \epsilon_{rst}\epsilon_{stq} L_r L_q = -\frac{\hbar}{2} C L^2$

$$E \to (2A - \hbar C)\hbar^2,\ (2A - \hbar C \pm B)\hbar^2$$

3 a) $\langle n=1|V|n=1\rangle = 0$ by parity

$\langle n=2|V|n=2\rangle = 0$ by parity

$\langle n=2\ \ell=1|V|n=2, \ell=0\rangle = 0$ Because $V \sim z\left[(x+iy)^2-(x-iy)^2\right]$ and m's differ by two units to be nonzero. Not possible

b)

$$\left(\begin{array}{c|ccc|ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & b & 0 & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\
\hline
0 & 0 & b^* & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d^* & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c^* & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$\ell=0 \qquad \ell=1 \qquad \ell=2$

c) $E = -\frac{me^4}{18\hbar^2}$ twice ($\ell=0$ and $\ell=2, m=0$)

for $\ell=1=m$ and $\ell=2, m=-1$

$$\begin{vmatrix} E & a \\ a^* & E \end{vmatrix} = 0 \quad \text{or} \quad E = -\frac{me^4}{18\hbar^2} \pm a$$

for $\ell=1, m=-1$ and $\ell=2\ m=1$

$$\begin{vmatrix} E & d \\ d^* & E \end{vmatrix} = 0 \quad \text{or} \quad E = -\frac{me^4}{18\hbar^2} \pm d$$

(which tells us that $|a|=|d|$)

$l=1,\ m=0\ ;\ l=2,\ m=\pm 2$

$$\begin{vmatrix} E & b & c \\ b^* & E & 0 \\ c^* & 0 & E \end{vmatrix} = 0$$

$$E\left(E^2 - |b|^2 - |c|^2\right) = 0$$

$$E = -\frac{me^4}{18\hbar^2} \pm \sqrt{|b|^2 + |c|^2} \quad \text{and} \quad -\frac{me^4}{18\hbar^2}$$

so a total of three unchanged levels

d) $V \sim z\left[(x+iy)^2 - (x-iy)^2\right]$

is an element of a spherical tensor of rank 3; $T^{(3)}_{\pm 2}$

$$\langle l=1, m|\ V\ |l=2\ m'\rangle = \langle l=1, n=3\,\|\,T^{(3)}\,\|\,l=2, n=3\rangle$$

$$\uparrow_{T^{(3)}_{\pm 2}} \qquad \langle l=1, m|\,3, \pm 2;\ 2\ m'\rangle$$

The Clebsch-Gordan coefficients are known so everything can be expressed in terms of a single parameter

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$$f(\theta) = -\frac{2m}{4\pi\hbar^2}\int V(r)\, e^{i(k_i - k_f)\cdot r}\, d^3r$$

$$= -\frac{mV_0}{2\pi\hbar^2}\, 2\pi\, \frac{1}{i|k_i - k_f|}\int_0^R r\, dr\ 2i\sin|k_i - k_f|\, r$$

$$= -\frac{2mV_0}{\hbar^2|k_i - k_f|}\frac{\partial}{\partial|k_i - k_f|}\int_0^R dr\left(-\cos|k_i - k_f|\, r\right)$$

$$= -\frac{2mV_0}{\hbar^2|k_i - k_f|}\frac{\partial}{\partial|k_i - k_f|}\frac{\sin|k_i - k_f|\, r}{|k_i - k_f|}$$

$$= -\frac{2mV_0}{\hbar^2|k_i - k_f|}\left[-\frac{\sin|k_i - k_f|R}{|k_i - k_f|^2} + R\frac{\cos|k_i - k_f|R}{|k_i - k_f|}\right]$$

$$\frac{d\sigma}{d\Omega} = |f|^2 = \left(\frac{2mV_0}{\hbar^2|k_i-k_f|^3}\right)^2 \left[\sin|k_i-k_f|R - R|k_i-k_f|\cos|k_i-k_f|R\right]^2$$

$$|k_i-k_f|^2 = 2k^2(1-\cos\theta)$$

$$= 4k^2\sin^2\theta/2$$

$$= \frac{8mE}{\hbar^2}\sin^2\theta/2$$