1. A particle of mass $m$ in three dimensions has Hamiltonian

$$H = \left[\frac{p_x^2}{2m} + V(x)\right] + \left[\frac{p_y^2}{2m} + V(y)\right] + \left[\frac{p_z^2}{2m} + V(z)\right]$$

where the $V$'s have the same infinite well form,

$$V(x) = \begin{cases} 0 & -\frac{L}{2} \le x \le \frac{L}{2} \\ \infty & \text{otherwise} \end{cases}$$

and similarly for $V(y)$ and $V(z)$.

a) What is a CSCO for this system?

b) Write the parity operator in terms of the CSCO.

c) Now include the perturbation

$$V'(x, y, z) = \lambda(xy + yz + xz).$$

What is the ground state energy to $O(\lambda)$?

d) For this potential what is the matrix which determines the energies of the lowest lying excited states to $O(\lambda)$? (You can use the result $\int_0^\pi \xi d\xi \cos(\xi/2) \sin\xi = \frac{16}{9}$.)

e) What are the eigenvalues corresponding to d)?

2. Four spin-1/2 particles with angular momenta $\hbar\sigma_i/2$ have Hamiltonian

$$H = \lambda(\sigma_1 \cdot \sigma_2)(\sigma_3 \cdot \sigma_4) + A(\sigma_{1x} + \sigma_{2x}) + B(\sigma_{3y} + \sigma_{4y})$$

where $\lambda, A$, and $B$ are all real constants.

a) Give a CSCO in which $H$ is diagonal.

b) Explain (verbally or mathematically) why your answer is correct.

c) Give all the eigenvalues of $H$.

d) Determine the trace of $H$ and show that it is consistent with the results of c).

3. A particle of mass $M$ is in the potential

$$V = \begin{cases} \infty & x < 0, x > L \\ -V_0\, \delta(x - L/2) & 0 < x < L \end{cases}$$

where $V_0 > 0$.

a) What are the energy levels and wave functions for $V_0 = 0$?

b) What are the energy levels to order $V_0$?

c) What are the energy levels to order $V_0^2$?

**Problem #4 is on the other side of the paper.**

4. A particle of charge $e$ and mass $M$ is confined by the harmonic potential

$$V(x) = \frac{1}{2}M[\omega_1^2(x^2 + y^2) + \omega_2^2 z^2].$$

where $\omega_1 > \omega_2$.

a) Assuming the particle to be in the first excited state determine the angular distribution of photons in the radiative decay process.

b) What is the polarization of the emitted radiation?

c) Prescribe a state whose decay to the ground state allows (in some direction) circularly polarized light to be detected.

1 a) $H_x, H_y, H_z$

where

$$H_x = \left(\frac{p_x^2}{2m} + V(x)\right) \ldots$$

b) $$E_{n_x n_y n_z} = \frac{\hbar^2 \pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2) \qquad n_x, n_y, n_z = 1, 2, \ldots$$

$$\mathbb{P} = \exp\left[\left(\frac{2mL^2}{\hbar^2\pi^2}\right)^{1/2}\left[(H_x)^{1/2} + (H_y)^{1/2} + (H_z)^{1/2}\right] \; i\pi\right]$$

c) $$(V')_{n_x = n_y = n_z = 1} = \left(\frac{2}{L}\right)^3 \iiint_{-L/2}^{L/2} \cos^2\frac{\pi x}{L} \cos^2\frac{\pi y}{L} \cos^2\frac{\pi z}{L}$$

$$\lambda\,(xy + yz + zx)\,dx\,dy\,dz \; = 0$$

d) Lowest excited states are $(2,1,1), (1,2,1), (1,1,2)$

$$V'/\lambda = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \left(\frac{2}{L}\right)^3 \frac{L}{2}\left[\int_{-L/2}^{L/2} x \cos\frac{\pi x}{L} \sin\frac{2\pi x}{L}\,dx\right]^2$$

$$[\quad]^2 = \left[\left(\frac{L}{2\pi}\right)^2 2 \int_0^{\pi} \xi\, d\xi \cos\xi/2 \sin\xi\right]^2$$

$$= \frac{L^4}{4\pi^4}\left(\frac{16}{9}\right)^2 = \frac{64L^4}{81\pi^4}$$

$$V' = \lambda \frac{256 L^2}{81\pi^4} = \lambda L^2 \left(\frac{4}{3}\right)^4 \frac{1}{\pi^4}$$

e) Need eigenvalues of

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} -\sigma & 1 & 1 \\ 1 & -\sigma & 1 \\ 1 & 1 & -\sigma \end{pmatrix} = -\sigma^3 + 2 + 3\sigma$$

Clearly $\sigma = -1$ is one root

$$-\sigma^3 + 2 + 3\sigma = (\sigma+1)(-\sigma^2 + A\sigma + B) = 0$$

$$= -\sigma^3 + A\sigma^2 + B\sigma - \sigma^2 + A\sigma + B$$

so $A = 1,\ B = 2$

$$(\sigma+1)(-\sigma^2 + \sigma + 2) = -(\sigma+1)(\sigma+1)(\sigma-2)$$

So $\sigma = -1, -1, 2$

$$H^{(2)} = \lambda L^2 \left(\frac{4}{3}\right)^4 \frac{1}{\pi^4} \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$$

Note trace $= 0$.

2 a) $\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, (\sigma_1+\sigma_2)^2, (\sigma_3+\sigma_4)^2, \sigma_{1x}+\sigma_{2x}, \sigma_{3y}+\sigma_{4y}$

b) $\sigma_i^2 = 3\mathbb{1}$ so all commutative. One can add angular momenta $\frac12\sigma_1$ and $\frac12\sigma_2$ to get $(\sigma_1+\sigma_2)^2$ (as usual) and any <u>one</u> component of $\sigma_1+\sigma_2$ can be diagonalized. So take $\sigma_{1x}+\sigma_{2x}$. Similarly for 3,4

c) $$\frac14 \sigma_1\cdot\sigma_2 = \frac{\frac14(\sigma_1+\sigma_2)^2 - \frac14\sigma_1^2 - \frac14\sigma_2^2}{2} = \frac{\binom{2}{0} - \frac32}{2} = \underset{s=1}{\frac14},\ \underset{s=0}{-\frac34}$$

c) $S_{12}=1,\ S_{34}=1$ (9 states)

$$E = \lambda + 2\begin{pmatrix} A+B \\ A \\ A-B \\ B \\ 0 \\ -B \\ -A+B \\ -A \\ -A-B \end{pmatrix}$$

$S_{12}=1,\ S_{34}=0$

$$E = -3\lambda + 2\begin{pmatrix} A \\ 0 \\ -A \end{pmatrix}$$

$S_{12}=0,\ S_{34}=1$

$$E = -3\lambda + 2\begin{pmatrix} B \\ 0 \\ -B \end{pmatrix}$$

$S_{12}=S_{34}=0$

$$E = 9\lambda$$

d) $\mathrm{Tr}\, H = 0$

$$\left.\begin{array}{ll} S_{12}=S_{34}=1 & \mathrm{Tr}\, H = 9\lambda \\ S_{12}=1, S_{34}=0 & \mathrm{Tr}\, H = -9\lambda \\ S_{12}=0, S_{34}=1 & \mathrm{Tr}\, H = -9\lambda \\ S_{12}=S_{34}=0 & \mathrm{Tr}\, H = 9\lambda \end{array}\right\}\ \text{sum is zero}$$

3. a) $E_n = \dfrac{\hbar^2 n^2 \pi^2}{2ML^2}$ $\qquad \psi_n = \sqrt{\dfrac{2}{L}}\sin\dfrac{n\pi x}{L}$

b) $$\langle n|H|n\rangle = \frac{\hbar^2 n^2 \pi^2}{2ML^2} - V_0 \sin^2\frac{n\pi}{2}\ \frac{2}{L}$$

$$= \frac{\hbar^2 n^2 \pi^2}{2ML^2} - \frac{2V_0}{L}\begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

c) $E_n^{(2)} = \sum{}' \dfrac{|\langle n|V|n'\rangle|^2}{E_n - E_{n'}}$

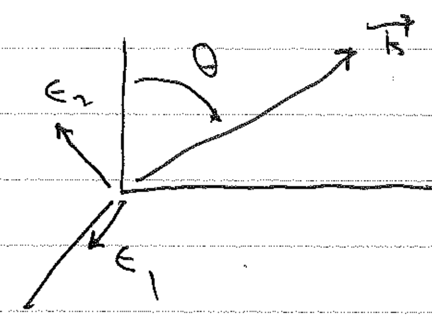
$= \dfrac{4V_0^2}{L^2} \sum{}' \dfrac{\sin^2\frac{n\pi}{2}\sin^2\frac{n'\pi}{2}}{\frac{\hbar^2\pi^2}{2mL^2}(n^2-n'^2)}$

$= \dfrac{8V_0^2 m}{\hbar^2\pi^2} \sum_{n'}{}' \dfrac{1}{n^2-(2n'+1)^2} \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

omit $2n'+1=n$ term

4 a) First excited state is

$E = \hbar\left[\omega_1 + \frac{3}{2}\omega_2\right]$




$p\cdot\epsilon \sim \sin\theta$

so $\sin^2\theta$ distribution

b) Polarization is linear and in plane of $\vec{k}$ and z-axis.

c) For particle in state $\sim (x \pm iy)e^{-\alpha^2x^2/2 - \beta^2y^2/2}$

$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|100\rangle \pm i|010\rangle\right]$

(i.e., $m=\pm1$) one has circular polarization along z-axis.

# 408 Final Solution Spring 15

## Problem 1

Solution:

**(a)** A CSCO would be $\{H_x, H_y, H_z\}$ with $H_x = \frac{p_x^2}{2m} + V(x)$ etc. If we write out the energy spectrum, it's $E(n_x, n_y, n_z) = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$, you notice that the combination $(n_x, n_y, n_z)$ uniquely defines each individual state (or lifts all the degeneracy).

**(b)** Once again, the energy levels are $E(n_x, n_y, n_z) = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$, with $n_x, n_y, n_z = 1, 2, 3....$ The possible values of each quantum number count for credits. For the 1-D case, take $x$ as our spatial coordinate. We know the parity of the wavefunction alternates based on $n_x$ is even or odd. Parity is even if $n_x$ is odd and vise versa. Therefore, the 1-D parity operator is going to be

$$\mathcal{P} = e^{i\pi(\frac{2mL^2}{\hbar^2\pi^2})^{\frac{1}{2}}(H_x)^{\frac{1}{2}}} = e^{i\pi n_x}$$

However, for even $n_x$ that operator gives eigenvalue $+1$, which means we get even parity. It's easy to fix just by multiplying $(-1)$. Therefore, the parity for 1-D case is

$$\mathcal{P} = (-1)e^{i\pi(\frac{2mL^2}{\hbar^2\pi^2})^{\frac{1}{2}}(H_x)^{\frac{1}{2}}}$$

Now let's go back to the 3-D case, the parity is

$$\mathcal{P} = (-1)e^{i\pi(\frac{2mL^2}{\hbar^2\pi^2})^{\frac{1}{2}}(H_x)^{\frac{1}{2}}}(-1)e^{i\pi(\frac{2mL^2}{\hbar^2\pi^2})^{\frac{1}{2}}(H_y)^{\frac{1}{2}}}(-1)e^{i\pi(\frac{2mL^2}{\hbar^2\pi^2})^{\frac{1}{2}}(H_z)^{\frac{1}{2}}} = (-1)e^{i\pi(\frac{2mL^2}{\hbar^2\pi^2})^{\frac{1}{2}}\{(H_x)^{\frac{1}{2}}+(H_y)^{\frac{1}{2}}+(H_z)^{\frac{1}{2}}\}}$$

Each time the quantum number of any spatial dimension is increased by one, the parity of the overall wavefunction changes.

Someone asked a question in lecture, should we always have the sqrt of the Hamiltonian. The answer is no. Take the 1-D SHO as an example. We know the parity of the wavefunction alternates as $n$ increases. The energy levels are $E_n = \hbar\omega(n + \frac{1}{2})$ for $n = 0, 1, 2...$ The parity operator including H in this case is

$$\mathcal{P} = e^{i\pi(\frac{H}{\hbar\omega} - \frac{1}{2})} = e^{i\pi n}$$

Here we don't need to fix anything since ground state is $n = 0$ with even parity.

**(c)** For ground state, $n_x = n_y = n_z = 0$, 1st order is trivially zero, since

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} xcos^2(\frac{\pi x}{L})dx \int_{-\frac{L}{2}}^{\frac{L}{2}} ycos^2(\frac{\pi y}{L})dy \int_{-\frac{L}{2}}^{\frac{L}{2}} dz = 0 \quad \text{for xy term, and same for the others}$$

**(d)(e)** solutions are pretty straightforward.

## Problem 2

Before we do anything, let's back up a little bit. When we coupled two spin particles, we mentioned that our CSCO choice could be $\{S_1^2, S_2^2, S_{1z}, S_{2z}\}$ in the uncoupled basis, or $\{(\vec{S_1} + \vec{S_2})^2, S_1^2, S_2^2, (\vec{S_1} + \vec{S_2})_z\}$

in the coupled basis. In the CSCO, there isn't any particular reason that we choose the z-component, instead of choosing x or y component. Therefore, I could write my CSCO as $\{S_1^2, S_2^2, S_{1x}, S_{2y}\}$ in the uncoupled basis or $\{(\vec{S_1} + \vec{S_2})^2, S_1^2, S_2^2, (\vec{S_1} + \vec{S_2})_x\}$ in the coupled basis.

When we add 3 spin particles, the number of operators in the CSCO is at least 6, since each particle needs its total spin and one of its component and we have 3 particles. In the uncoupled basis, one CSCO choice would be $\{S_1^2, S_2^2, S_3^2, S_{1z}, S_{2z}, S_{3z}\}$, while in the coupled basis a CSCO is $\{(\vec{S_1} + \vec{S_2} + \vec{S_3})^2, S_1^2, S_2^2, S_3^2, (\vec{S_1} + \vec{S_2})^2, (\vec{S_1} + \vec{S_2} + \vec{S_3})_z\}$. For the coupled one, recall that we add the first two spins together, and add the third one to the result we get from coupling the first two. Once again, we could choose any component of the spin in our CSCO.

Now we are adding four spins together. CSCO in uncoupled basis is $\{S_1^2, S_2^2, S_3^2, S_4^2, S_{1z}, S_{2z}, S_{3z}, S_{4z}\}$. In the coupled basis, we could add the 1st and 2nd together, 3rd and 4th together. Therefore, a CSCO choice would be $\{(\vec{S_1} + \vec{S_2})^2, S_1^2, S_2^2, (\vec{S_1} + \vec{S_2})_z, (\vec{S_3} + \vec{S_4})^2, S_3^2, S_4^2, (\vec{S_3} + \vec{S_4})_z\}$. You could change the component if you'd like to.

**(a)(b)** Let's get back to the problem. If you are only asked to give a CSCO, both ones up there would work. However, here you want H to be diagonal in your CSCO basis. The reason that $\{(\vec{S_1} + \vec{S_2})^2, S_1^2, S_2^2, (\vec{S_1} + \vec{S_2})_z, (\vec{S_3} + \vec{S_4})^2, S_3^2, S_4^2, (\vec{S_3} + \vec{S_4})_z\}$ doesn't work is $(\vec{S_1} + \vec{S_2})_z, (\vec{S_3} + \vec{S_4})_z$ don't diagonalize H. Recall in the Pauli Spin matrices, if $H = AS_x$, H is only diagonalized in $S_x$ eigenbasis, but not in $S_z$ eigenbasis. Same situation here, based on the Hamiltonian given in this problem, we choose our CSCO to be $\{(\vec{S_1} + \vec{S_2})^2, S_1^2, S_2^2, (\vec{S_1} + \vec{S_2})_x, (\vec{S_3} + \vec{S_4})^2, S_3^2, S_4^2, (\vec{S_3} + \vec{S_4})_y\}$ so that H is diagonal. You also need to mention that they trivially commute to each other to get full credits.

The rest of parts are pretty clear based on the solutions.