Physics 408 Midterm March 3, 2016

1. Positronium is (as you know) the bound state of an electron and a positron. The lowest state is split by a spin-spin coupling.

$$V_1 = \eta(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \qquad\qquad \eta = \frac{7}{48}\alpha^4 mc^2$$

where $\frac{1}{2}\,\boldsymbol{\sigma}^{(1)}$ and $\frac{1}{2}\,\boldsymbol{\sigma}^{(2)}$ are spin operators for electron and positron respectively. A magnetic field B in the z-direction adds a term

$$V_2 = \mu_B B\left(\sigma_z^{(1)} - \sigma_z^{(2)}\right)$$

to the Hamiltonian.

a) Give the matrix of $V_1 + V_2$ in an appropriate basis.

b) What are the eigenvalues of H?

c) Find the exact eigenstates of H in terms of the set $|S, M>$ (which has the usual meaning).

d) Calculate the trace of H and relate it to the results of b).

e) If positronium decays into photons, what is the total energy available to the photons as measured in the rest frame of the positronium?

2. a) What are the commutation relations of an angular momentum $\mathbf{J}$? Give your answers in terms of J_z, $J_\pm$.

b) Give the result of J_z and $J_\pm$ acting on eigenstates $|J, J_z>$.

c) Construct the states of total angular momentum J for two particles of angular momentum one.

d) In terms of the operator J^2 give the explicit form of the exchange operator.

e) Compute the matrix elements

$$< JM|\mathrm{r}|JM' >$$

for general J in terms of a single parameter.

1 a) $\frac{1}{4}\sigma_1\cdot\sigma_2 = \frac{j(j+1)-3/2}{2} = \frac{(2,0)-3/2}{2}$

$$\sigma_1\cdot\sigma_2 = 2\left[(2,0)-3/2\right]$$
$$= (1,-3)$$

$\langle S M|\sigma_z^{(1)}-\sigma_z^{(2)}|S'M'\rangle \neq 0$ only if $S\neq S'$ (exchange symmetry) also $M=M'$ to be nonzero. Thus only

$\langle S=1, M=0|\sigma_z^{(1)}-\sigma_z^{(2)}|S=0\ M=0\rangle$ (and c.c.)

$$\frac{1}{2}\left(\langle \tfrac12\,-\tfrac12| + \langle -\tfrac12\,\tfrac12|\right)\underbrace{(\sigma_z^{(1)}-\sigma_z^{(2)})\left(|\tfrac12\,-\tfrac12\rangle - |-\tfrac12\,\tfrac12\rangle\right)}_{2\left(|\frac12\,-\frac12\rangle+|-\frac12\,\frac12\rangle\right)}$$

$$= 2$$

$$\langle V_1+V_2\rangle = \begin{array}{c} \begin{array}{cccc} S=0 & \overbrace{M=1 \quad 0 \quad -1}^{S=1} \end{array} \\ \begin{pmatrix} -3\eta & 0 & 2\mu_B B & 0 \\ 0 & \eta & 0 & 0 \\ 2\mu_B B & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \end{pmatrix} \end{array}$$

b) Eigenvalues are η, η and $\det\begin{pmatrix} -3\eta-\lambda & 2\mu_B B \\ 2\mu_B B & \eta-\lambda \end{pmatrix}$

$$\lambda^2 + 2\lambda\eta - 3\eta^2 - 4\mu_B^2 B^2 = 0$$

$$\lambda = -\eta \pm \sqrt{4\eta^2+4\mu_B^2B^2} = -\eta \pm 2\sqrt{\eta^2+\mu_B^2B^2}$$

c) Two eigenstates are easy $|\tfrac12\,\tfrac12\rangle$ and $|-\tfrac12\,-\tfrac12\rangle$

The others $H\left[\alpha|10\rangle+\beta|00\rangle\right] = \alpha\eta|10\rangle - 3\eta|00\rangle\beta$

$$+2\mu_B B\left(\alpha|00\rangle+\beta|10\rangle\right)$$
$$= E\left[\alpha|10\rangle+\beta|00\rangle\right]$$

for $E\alpha = \eta\alpha + 2\mu_B B\beta$

$E\beta = -3\eta\beta + 2\mu_B B\alpha$

Look at $E = -\eta \pm 2\sqrt{}$

$$\alpha\left[-2\eta - 2\sqrt{}\right] = 2\mu_B B\beta$$

$$\alpha = -\frac{\mu_B B}{\eta + \sqrt{}}\beta \text{ and } \alpha^2 + \beta^2 = 1$$

$$\beta^2\left(1 + \frac{\mu_B^2 B^2}{(\eta + \sqrt{})^2}\right) = 1$$

$$\beta^2 = \left(\frac{2\eta^2 + 2\mu^2 B^2 + 2\eta\sqrt{}}{(\eta + \sqrt{})^2}\right)^{-1}$$

$$\beta = \frac{\eta + \sqrt{}}{\sqrt{2}\left(\sqrt{}^2 + \eta\sqrt{}\right)^{1/2}} = \frac{1}{\sqrt{2}}\frac{1 + \eta/\sqrt{}}{(1 + \eta/\sqrt{})^{1/2}}$$

$$= \frac{1}{\sqrt{2}}\left(1 + \frac{\eta}{\sqrt{}}\right)^{1/2} \Rightarrow \alpha = \frac{-1}{\sqrt{2}}\left(1 - \frac{\eta}{\sqrt{}}\right)^{1/2}$$

get second by $\alpha \to \beta$, $\beta \to -\alpha$

d) $\mathrm{Tr}\, H = 0$ in agreement with $\eta, \eta, -\eta \pm 2\sqrt{}$

e) $$2mc^2 - \frac{\frac{1}{4}me^4}{2\hbar^2} + \begin{pmatrix} -\eta + \sqrt{} \\ -\eta - \sqrt{} \\ \eta \end{pmatrix} = \text{sum of photon energies}$$

(not reduced mass!)

2. a) $[J_x, J_y] = i\hbar J_z$ etc

$$J_\pm = J_x \pm i J_y$$

$$[J_+, J_-] = 2\hbar J_z$$

$$[J_\pm, J_z] = [J_x \pm i J_y, J_z] = \hbar\left[-i J_y \pm (-1) J_x\right]$$

$$= \mp\hbar\left[J_x \pm i J_y\right] = \mp\hbar J_\pm$$

b) $$J_z|J J_z\rangle = \hbar J_z|J J_z\rangle$$

$$J_\pm|J J_z\rangle = \hbar\left[(J \mp J_z)(J \pm J_z + 1)\right]^{1/2}|J J_z\rangle$$

c) $|22\rangle = |11\rangle|11\rangle$

$2|21\rangle = \sqrt{2}\left[|10\rangle|11\rangle + |11\rangle|10\rangle\right]$

$|21\rangle = \frac{1}{\sqrt{2}}\left[|10\rangle|11\rangle + |11\rangle|10\rangle\right]$

$\sqrt{6}\,|20\rangle = \frac{1}{\sqrt{2}}\Big[\sqrt{2}\,|1-1\rangle|11\rangle + \sqrt{2}\,|10\rangle|10\rangle$

$+\sqrt{2}\,|10\rangle|10\rangle + \sqrt{2}\,|11\rangle|1-1\rangle\Big]$

$|20\rangle = \frac{1}{\sqrt{6}}\left[|1-1\rangle|11\rangle + |11\rangle|1-1\rangle + 2|10\rangle|10\rangle\right]$

$|2-1\rangle = \frac{1}{\sqrt{2}}\left[|10\rangle|1-1\rangle + |1-1\rangle|10\rangle\right]$

$|2-2\rangle = |1-1\rangle|1-1\rangle$

$|11\rangle = \frac{1}{\sqrt{2}}\left[|11\rangle|10\rangle - |10\rangle|11\rangle\right]$

$\sqrt{2}\,|10\rangle = \frac{1}{\sqrt{2}}\Big[\cancel{\sqrt{2}\,|10\rangle|10\rangle} + \sqrt{2}\,|11\rangle|1-1\rangle - \sqrt{2}\,|1-1\rangle|11\rangle$

$\cancel{-\sqrt{2}\,|10\rangle|10\rangle}\Big]$

$|10\rangle = \frac{1}{\sqrt{2}}\left[|11\rangle|1-1\rangle - |1-1\rangle|11\rangle\right]$

$|1-1\rangle = \frac{1}{\sqrt{2}}\left[|10\rangle|1-1\rangle - |1-1\rangle|10\rangle\right]$

$|00\rangle = A|11\rangle|1-1\rangle + B|10\rangle|10\rangle + C|1-1\rangle|11\rangle$

$A - C = 0$

$2B + A + C = 0$ or $B = -A$

$A = \sqrt{\frac{1}{3}}$

$|00\rangle = \frac{1}{\sqrt{3}}\left[|11\rangle|1-1\rangle - |10\rangle|10\rangle + |1-1\rangle|11\rangle\right]$

d) $P_2 = \dfrac{(J^2 - 2\hbar^2)J^2}{24\hbar^4}$

$P_1 = \dfrac{J^2(6\hbar^2 - J^2)}{8\hbar^4}$

$P_0 = \dfrac{(2\hbar^2 - J^2)(6\hbar^2 - J^2)}{12\hbar^4}$

$$P_0 + P_1 + P_2 = 1 - \frac{2}{3}\frac{J^2}{\hbar^2} + \frac{1}{12}\frac{J^4}{\hbar^4} + \frac{3}{4}\frac{J^2}{\hbar^2} - \frac{J^4}{8\hbar^4} + \frac{J^4}{24\hbar^4} - \frac{J^2}{12\hbar^2}$$

$$= 1$$

$$X = P_2 + P_0 - P_1$$

$$= 1 - \frac{2}{3}J^2/\hbar^2 + \frac{J^4}{12\hbar^4} - \frac{3}{4}\frac{J^2}{\hbar^2} + \frac{J^4}{8\hbar^4} + \frac{J^4}{24\hbar^4} - \frac{J^2}{12\hbar^2}$$

$$= 1 - \frac{3}{2}\frac{J^2}{\hbar^2} + \frac{J^4}{4\hbar^4}$$

e) $\langle JM|\vec{r}\,|JM'\rangle = \dfrac{\alpha}{\hbar}\langle JM|\,J\,|JM'\rangle$

$$\langle JM|z|JM'\rangle = \alpha M\,\delta_{MM'}$$

$$\langle JM|x|JM'\rangle = \frac{\alpha}{2}\Big[\sqrt{(J+M')(J-M'+1)}\,\delta_{M\,M'+1} + \sqrt{(J-M')(J+M'+1)}\,\delta_{M\,M'-1}\Big]$$

$$\langle JM|y|JM'\rangle = \frac{\alpha}{2i}\Big[\sqrt{(J+M')(J-M'+1)}\,\delta_{M\,M'+1} - \sqrt{(J-M')(J+M'+1)}\,\delta_{M\,M'-1}\Big]$$