PHY 121 Mechanics *S.G. Rajeev*

Lecture 11

Conservation of Momentum

•The *momentum* $\mathbf{p}$ of a particle of mass m and velocity $\mathbf{v}$ is defined to be

$$\mathbf{p} = m\mathbf{v}.$$

•Newton's second law states that

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}$$

where $\mathbf{F}$ is the total force acting on the particle.

•Consider two particles of masses m_1 and m_2 and velocities $\mathbf{v_1}$ and $\mathbf{v_2}$. Their total momentum

is the vector sum

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$$

•Let $\mathbf{F}_{12}$ be the force that particle 1 exerts on particle 2 and $\mathbf{F}_{21}$ that exerted by particle 2 on particle 1. Suppose that the two particles are isolated from the rest of the world. Newton's third law states that

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

•Thus

$$\frac{d\mathbf{p_1}}{dt} = \mathbf{F}_{12}, \quad \frac{d\mathbf{p_2}}{dt} = \mathbf{F}_{21}$$

Combining the second and third laws

of motion gives

$$\frac{d\mathbf{p_1}}{dt} = -\frac{d\mathbf{p_2}}{dt}$$

or that

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

This is an example of a basic law of physics, *the law of conservation of momentum*:

The total momentum of an isolated system of particles is a constant.

•Notice that it is not the sum of the velocities that is conserved. The total velocity of a system of particles has no particular physical meaning.

Recoil of a Cannon

Imagine that a cannon of mass M is mounted on frictionless rails, so that it can be viewed as an isolated system. It is initially at rest. It fires a cannonball of mass m, in a horizontal direction, at a velocity of v *relative to the Earth*. The cannon will recoil at a velocity V. The initial momentum was zero. The total momentum of the cannon and the ball should be zero after the firing. Thus

$$MV + mv = 0$$

so that the recoil velocity of the cannon is

$$V = -\frac{m}{M}v.$$

The heavier the cannon the slower it will recoil.

•If the velocity of the ball were measured relative to the cannon the answer would be slightly different. Let $\mathbf{u}$ be the muzzle velocity of the cannon; i.e., with respect to the cannon. Then the velocity of the ball relative to the earth will be

$$v = u + V.$$

Then

$$MV + m(u + V) = 0$$

and

$$V = -\frac{m}{m+M}u.$$

Since the cannon is usually much heavier than than the ball, $m+M$ and M are almost the same.

•What if the ball is fired at an angle θ to the horizontal? The vertical component of the momentum of the cannon and ball is not conserved since we cannot ignore the momentum of the earth. But the horizontal component is conserved. We will get

$$MV + m(u\cos\theta + V) = 0$$

so that

$$V = -\frac{m}{m+M}u\cos\theta.$$

For example if the ball is fired vertically upward the cannon doesn't recoil at all.

Systems with Variable Mass

•Occassionally we have to deal with systems of varying mass. A rocket is a good example: it loses mass as burnt fuel is expelled. The law of conservation of momentum is a good tool in understanding how such systems move.

•Imagine a rocket of initial mass M is ejecting exhaust gases with a speed u relative to the rocket, losing mass at a constant rate μ per second. It is obvious that the acceleration of the rocket will grow with time, until it runs out of fuel. Its mass at time t is

$$M(t) = M - \mu t.$$

•Imagine that the rocket is moving in a straight line, in zero gravity. It starts at rest. Let $v(t)$ be its velocity at time t. In a small time interval dt, its momentum changes by

$$\begin{aligned} dP(t) &= d[M(t)v(t)] \\ &= dM(t)v(t) + M(t)dv(t). \end{aligned}$$

The velocity of the exhaust relative to the rocket is $-u$, so relative to an inertial observer it is

$$-u + v(t)$$

and momentum of the exhaust emitted in the time interval dt is

$$\mu dt(-u + v(t)).$$

By conservation of momentum,

$dM(t)v(t) + M(t)dv(t)\ +$

$$\mu dt(-u + v(t)) = 0$$

Thus

$$M(t)\frac{dv(t)}{dt} = \mu u$$

or

$$dv(t) = \frac{\mu u}{M - \mu t}dt.$$

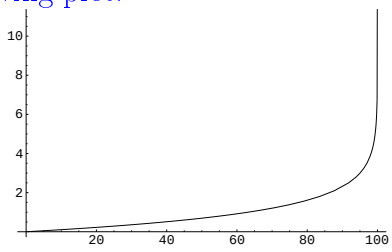
This is an example of a *differential equation*. We get

$$v(t) = \int_0^t \frac{\mu u}{M - \mu t}dt$$

or,

$$v(t) = u \ln[\frac{M}{M - \mu t}].$$

For this answer to make sense, $M - \mu t$ must be positive; the mass of the rocket cannot become negative.

- For a rocket of inital mass of $M = 1000$ tonnes, losing mass at a rate

of $\mu = 10$ tonnes per second and an exhaust speed of $u = 1$ km/sec the velocity will be given by the following plot: