PHY 121 Mechanics *S.G. Rajeev*

Lecture 12

Center of Mass

•Consider two particles of masses m_1 and m_2 and momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. They need *not* be isolated from the rest of the world. Let $\mathbf{F}_1$ be the total force on the first particle and $\mathbf{F}_2$ that on the second. Clearly,

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = \mathbf{F}_1 + \mathbf{F}_2.$$

•Thus the two bodies move as though they had a total force $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ acting on it; the total mass is $m = m_1 + m_2$. The total momentum is

$$\mathbf{p} = m_1 \frac{d}{dt}\mathbf{r}_1 + m_2 \frac{d}{dt}\mathbf{r}_2$$

•It is as if the total mass of the system is concentrated at a point $\mathbf{r}_{cm}$,

$$\mathbf{p} = m \frac{d}{dt}\mathbf{r}_{cm}$$

where

$$\mathbf{r}_{cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}.$$

This point is called the *center of mass* of this pair of particles.

•Thus we can replace the collective motion of a system of particles by the motion of its center of mass. This is a major simplification: even if a body is made up of

many particles, its overall motion can be described as though it were a single particle located at its center of mass.

•If we have many particles of masses $m_1, m_2, \cdots$ and positions $\mathbf{r}_1, \mathbf{r}_2, \cdots$ the center of mass is given by

$$\mathbf{r}_{cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots}{m_1 + m_2 + \cdots}.$$

Rigid Body

•In fact this idea applies even in the limit when a body is made of an infinite number of particles. An example of this is a *rigid body*, which is made of an infinite number of particles which are required to move together: they all have the same velocity and acceleration.

•The mass Δm of any small volume ΔV will be $\rho\Delta V$ where ρ is the *density* of the body. Density may not be a constant: it can vary from point to point. The total mass will be the integral

$$m = \int_V \rho(\mathbf{r})dxdydz$$

where V is the volume of space occupied by the body.

•The center of mass of the rigid body will be at

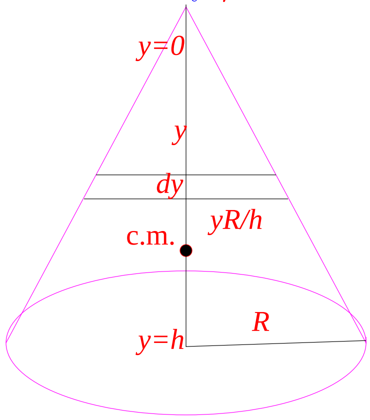
$$\mathbf{r}_{cm} = \frac{1}{m} \int_V \mathbf{r}\rho(\mathbf{r}) dxdydz$$

•There is a way to think of the center of mass of a rigid body using its own weight. The point at which we have to support the body so that will be balanced is the center of mass.

•Often we can tell where the center of mass of a body is from the symmetry of its shape. The center of mass of a uniform rod will be it at its center. The center of mass of a uniform sphere will be at its center. Sometimes we have to do an integral to get the center of mass.

Center of Mass of a Cone

•Consider a right circular cone of height h and radius R at the base. Assume the density ρ is uniform.



•By the symmetry of rotation around the vertical axis, we can see that the center of mass is somewhere along this axis. Also it must be closer

to the bottom than the top. But where exactly is it?

•Think of a little slice of the cone at a distance of y *from the top* and thinkness dy . It is almost a circular cylinder of radius $r = \frac{yR}{h}$. Its volume is thus $dV = \pi r^2 dy$.

•The mass in this little slice is ρdV . The total mass is

$$m = \int_0^h \pi (\frac{yR}{h})^2 \rho dy$$

$$= \pi \rho (\frac{R^2}{h^2}) \int_0^h y^2 dy$$

•The center of mass will be given by

$$y_{cm} = \frac{1}{m} \int_0^h \pi (\frac{yR}{h})^2 \rho y dy$$

$$= \frac{1}{m} \pi \rho (\frac{R^2}{h^2}) \int_0^h y^3 dy$$

•All the constant factors such as π^2 and R^2 cancel out so that we have

$$y_{cm} = \frac{\int_0^h y^3 dy}{\int_0^h y^2 dy}$$

•We can now use the integrals

$$\int_0^h y^2 dy = \frac{h^3}{3}$$

and

$$\int_0^h y^3 dy = \frac{h^4}{4}$$

to get

$$y_{cm} = \frac{3}{4} h.$$

Thus the center of mass of the cone is three-quarters of the way down the central axis.

Center of Mass of a Rocket

- Imagine a rocket that has the shape of a cone sitting on top of a cylinder of height L. Assume that the cylinder has a mass of M and the cone a mass of m. The c.m. of the cylinder will be at a height of $\frac{L}{2}$ and the cone at a height of $L+\frac{1}{4}h$. Thus the center of mass of the rocket will be at a height

$$\frac{M\frac{L}{2}+m(L+\frac{h}{4})}{M+m}$$