PHY 121 Mechanics *S.G. Rajeev*

Lecture 15

Angular Momentum and Torque

•The *angular momentum* of a particle with respect to a point is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where $\mathbf{r}$ is the position vector and $\mathbf{p}$ the momentum.

•The *torque* acting on a particle is

$$\tau = \mathbf{r} \times \mathbf{F}$$

where $\mathbf{r}$ is the position of the point at which the force $\mathbf{F}$ is applied.

•The total angular momentum of a body is the vector sum of the angular momenta of its component parts.

•Newton's second law for angular motion:

$$\frac{d\mathbf{L}}{dt} = \tau$$

The rate of change of angular momentum is equal to the torque.

•This is not a new law of physics, but rather is a consequence of the usual second law of motion of Newton. Indeed,

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$
$$= \mathbf{r} \times \mathbf{F}.$$

•Since velocity and momentum are parallel, their cross product is zero.

Conservation of Angular Momentum

•Analogue of Newton's third law of motion: *the torque exerted on body A by body B is equal in magnitude, but opposite in direction, to the torque exerted on B by A.*

•Again, this is not an independent law of physics, but can be derived from the usual third law of Newton.

•It follows then that the total angular momentum of a pair of isolated bodies is a constant in time. More generally *the total angular momentum of an isolated system is conserved.*

Why do some helicopters have a little rotor at the tail?

•Imagine a helicopter which has been lifted to an isolated point far away outside the Earth's atmosphere. To begin with it is at rest and then we turn on the engine. (It is an electric engine so it works outside the atmosphere). The blades of the helicopter will turn. Conservation of angular momentum will require that the rest of the helicopter turns in the opposite direction.

•Within the atmosphere, we can prevent this from happening by applying an external torque on the body of the helicopter: this is the purpose of the little rotor at the tail of a helicopter. It pushes against the air creating the torque necessary to prevent the helicopter from turning. It is put as far away from the axis as possible so that as much torque as possible can be produced with a given force.

•Another solution would be to have two sets of rotor blades mounted on the same axis but rotating in oppsite directions. Or there could be two sets mounted at opposite ends rotating in the same plane. *Should they be rotating in the same direction or opposite directions?*

•Even a single engine propeller plane has a tendency to turn around the horizontal axis. One of the wings produces slightly more lift than the

other to counteract this tendency.

•A bullet is made to spin on its axis as it leaves the muzzle; that is the purpose of 'rifling'. This makes it more stable as conservation of angular momentum requires the axis of the spin to remain conserved. This is slo why football players make the ball spin as it is thrown.

•Why is a bicycle more stable when it is moving than at rest? The angular momentum of the wheels needs to be conserved, so small forces won't topple the bicycle if its wheels are turning.

Rigid Bodies

•A *rigid body* is an object all of whose component parts move together: the distance between any two points of the body is constant in time.

•The translational motion of the rigid body is described by the motion of its center of mass. Even if the center of mass is fixed, the body can still move by *rotation*.

•Clearly the velocity of a particle inside a rotating rigid body depends on the distance from the axis: the velocity grows as we move away from the axis.

•All the particles making up the body move with a common *an-*

gular velocity; they move through the same angle in a given interval of time. If we know the angular position $\theta(t)$ at any instant in time t, the angular velocity is just the derivative:

$$\omega(t) = \frac{d}{dt}\theta(t)$$

•The simplest case of rotation is when the angular velocity is constant. But in general angular velocity can depend on time. *Angular acceleration* is the derivative of angular velocity with respect to time; or, it is the second derivative of angular position.

•Angular velocity is a vector; it points along the axis of rotation. If we curl the fingers of our right hand along the direction of rotation, the thumb points in the direction of angular velocity.

•A body can have kinetic energy due to rotation, in addition to the translational kinetic energy of its center of mass. We can calculate this by adding up the kinetic energies of all the particles that make up the body.

•Suppose a rigid body rotates around the z axis with angular velocity ω . Consider a small volume of the body at a co-ordinate (x, y, z) . The mass of this volume will be

$$dm = \rho(\mathbf{r})dxdydz.$$

•Its speed will be

$$v = \omega\sqrt{(x^2 + y^2)}$$

since $\sqrt{(x^2 + y^2)}$ is the distance from the axis.

•Thus the rotational kinetic energy of this small piece will be

$$\begin{aligned} dK &= \frac{1}{2}dm\omega^2(x^2 + y^2) \\ &= \frac{1}{2}\rho(\mathbf{r})(x^2 + y^2)dxdydz \;\; \omega^2 \end{aligned}$$

•The total rotational kinetic energy due to rotation around the z -axis will be

$$K_{\text{rot}} = \frac{1}{2}[\int \rho(\mathbf{r})(x^2 + y^2)dxdydz] \;\; \omega^2$$

•Notice that this is very similar to the expression for translational

kinetic energy , $K_{\text{trans}} = \frac{1}{2}mv^2$. Thus there is a quantity that describes the inertia of rotation, the rotational analogue of mass. This quantity is called the moment of inertia. The *moment of inertia* around the z -axis is

$$I_z = \int \rho(\mathbf{r})[x^2 + y^2]dxdydz.$$

•Indeed there are rotational analogues for *momentum*, *force*, and even *Newton's second law* of motion.