PHY 121 Mechanics *S.G. Rajeev*

Lecture 16

Moment of Inertia

•Let us summarize the analogies between rotational and translational motion:

position	*angular position*	$\theta(t)$
velocity	*angular velocity*	$\omega = \frac{d\theta(t)}{dt}$
acceleration	*angular acceleration*	$\frac{d^2\theta(t)}{dt^2}$
mass	*moment of inertia*	I
momentum	*angular momentum*	$L = I\omega$
force	*Torque*	$\tau = \mathbf{r} \times \mathbf{F}$

•The analogue of momentum is *angular momentum*, the product of moment of inertia and angular

velocity. Angular momentum, like angular velocity is a vector.

•The moment of inertia depends on the axis of rotation: it is not a scalar quantity. It is in fact a *tensor*, a quantity that is analogous to a *matrix*. Recall that a matrix acts on a vector to give you another vector. Moment of inertia acts on angular velocity and produces angular momentum.

Moment of Inertia of some Bodies of Constant Density

•The moment of inertia, around the central axis, of a thin hoop of mass m and radius R is just mR^2 . For, all particles are at the same distance R from the axis.

•The moment of inertia of a cylinder (of mass m , radius R and length L) around its central axis is $\frac{1}{2}mR^2$. For it can be thought of as made of many thin hoops of radii r and mass $\rho 2\pi r L$. Thus moment of inertia is

$$I = \int_0^R \rho 2\pi r L r^2 dr = \frac{2}{4}\rho\pi L R^4 = \frac{1}{2}mR^2$$

since

$$m = \rho\pi R^2 L.$$

Rolling

•A wheeel has both translational and rotational kinetic energy. If it has mass m , moment of inertia I , radius R and angular velocity ω , the translational kinetic energy is

$$K_{\text{trans}} = \frac{1}{2}mR^2\omega^2$$

and the rotaional kinetic energy is

$$K_{\text{rot}} = \frac{1}{2}I\omega^2.$$

•For example a circular hoop has $I = MR^2$ and

$$K_{\text{tot}} = MR^2\omega^2 = Mv^2.$$

•A disk has $I = \frac{1}{2}MR^2$ so that

$$K_{\rm tot} = \frac{1}{2} + \frac{1}{4}MR^2\omega^2 = \frac{3}{4}Mv^2$$

•A sphere has $I = \frac{2}{5}MR^2$ and

$$K_{\rm tot} = \frac{1}{2} + \frac{1}{5}MR^2\omega^2 = \frac{7}{10}Mv^2$$

Advanced Topic: Moment of Inertia is a Tensor

•the analogy between mass and moment of inertia is not perfect. Mass is scalar: momentum and velocity are always parallel to each other as vectors.

• *In general angular momentum and angular velocity are not parallel to each other.* Moment of inertia is a *tensor* (or *matrix*), an object with nine components.

•A matrix (or tensor) can be thought of as a square array of real numbers much as a vector can be thought of as a row (or column) of real numbers. There is a way to multiply a vector by a matrix to get another

vector:

$$\begin{pmatrix} v_x & v_y & v_z \end{pmatrix} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

means

$$\begin{aligned} v_x &= A_{xx}u_x + A_{xy}u_y + A_{xz}u_z \\ v_y &= A_{yx}u_x + A_{yy}u_y + A_{yz}u_z \\ v_z &= A_{zx}u_x + A_{zy}u_y + A_{zz}u_z \end{aligned}$$

•There is also a way of multiplying two matrices to get another matrix, but we don't need that here.

•Now, moment of inertia is a matrix and the general formula for angular momentum in terms of angular velocity is

$$\begin{pmatrix} L_x & L_y & L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

•It is always a *symmetric matrix*: $I_{xy} = I_{yx}$ etc. Sometimes it is a diagonal matrix, meaning that only I_{xx}, I_{yy}, I_{zz} are non-zero.

Formula for the Moment of Inertia Tensor

- The formula for the moment of inertia is

$$I_{xx} = \int \rho(\mathbf{r})[y^2 + z^2]dxdydz$$

$$I_{yy} = \int \rho(\mathbf{r})[z^2 + x^2]dxdydz$$

$$I_{zz} = \int \rho(\mathbf{r})[x^2 + y^2]dxdydz$$

$$I_{xy} = \int \rho(\mathbf{r})xydxdydz$$

$$I_{yz} = \int \rho(\mathbf{r})yzdxdydz$$

$$I_{zx} = \int \rho(\mathbf{r})zxdxdydz$$

- In the simplest case of an isotropic body (eg., a sphere) the moment of inertia is a multiple of the identity matrix:

$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

- In this case the angular momentum and angular velocity are parallel and moment of inertia behaves like a scalar. But this is not often the case.
- The general theory of the motion of a rigid body is one of the gems of mathematical physics. Along with celestial mechanics, it remains as the best example of a solvable yet complex dynamical system.