PHY 121 Mechanics *S.G. Rajeev*

Lecture 17

Kepler's Laws of Planetary Motion

•From the beginning of human civilization, people have always wondered about the motion of celestial bodies. There were a large number of 'fixed' stars that rise and set everyday. We now understand that this is due to the rotation of the Earth around its axis.

•Then there are the five 'wanderers' or 'planets' which seem to have an erratic motion relative to the fixed stars:Mercury, Venus, Mars, Jupiter and Saturn. Astrology was the first attempt to make sense of their motion.

•Planetary motion is very complicated; for example Venus and Mecury seem to reverse direction in the sky occassionally.

•But patterns in this motion began to emerge already in ancient times. For example, Jupiter's motion seemed to repeat itself every twelve years.

•It was thought that planets have a basic circular motion around the Earth every twenty-four hours. Further circular motions of periods of a few years were superimposed on this one. Very complicated schemes were devised to explain the enormous amount of observations.

•Starting with Aristarchus there was a view that this whole picture would be simplified if instead we assumed that all planets revolve around the Sun in *circles*. The Earth itself is such a planet. Moreover the Earth rotates around its axis once a day.

•Copernicus revived this idea in the Europe where it was forgotten for more than a thousand years. It was never forgotten in Arab, Indian and Chinese cultures, and was widely used all through the middle ages. (In fact, Copernicus refers to a Latin translation of an Arabic work on Aristarchus as the source of the idea.)

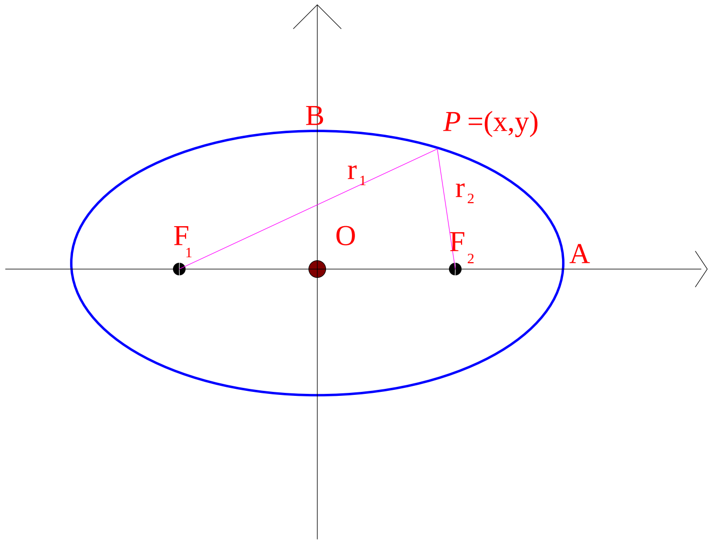
•This much simpler picture correctly explained much of the data. But there were still some small discrepancies.

•Kepler used the very accurate measurements of Tycho Brahe to make the next major advance. He noted that if the planets were moving in ellipses instead of circles, the discrepancies disappeared. Moreover he found some regularities in the planetary motion, which he codified into three laws.

What is an Ellipse?

- A *circle* with *center* O is the set of points at a *fixed distance* from O.
- An *ellipse* with *Foci* F_1, F_2 is the set of points for which the *sum of the distances to the foci* is fixed.
- The distance $a = OA$ is called the *semi-major axis*; $b = OB$ is the *semi-minor axis*. The distance of the focii to the origin is $OF_2 = F_1O = a - b$.
- If $a = b$ the ellipse becomes a circle. The *eccentricity* is $e = \frac{a-b}{a}$; so it measures how much an ellipse departs from being a circle.

The Ellipse



The sum of the distances of P to the Focii is a constant

Equation for an Ellipse

- We can also think of a circle (of radius a and center at the origin) as the set of points $P = (x, y)$ satisfying

$$x^2 + y^2 = a^2.$$

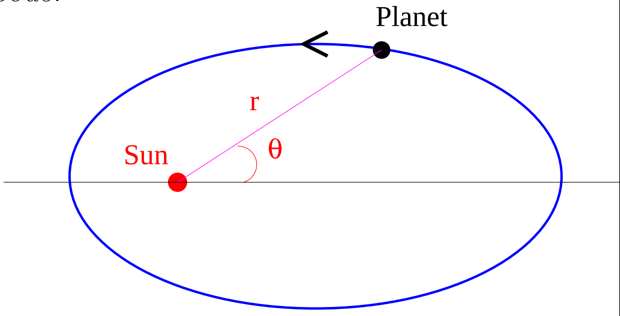
- In an analogous way an ellipse with center at the origin and semi-major axis a, semi-minor axis b is the set of points $P = (x, y)$ satisfying

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

- A challenge: prove that two definitions of the ellipse are equivalent.

Kepler's First Law of Planetary Motion

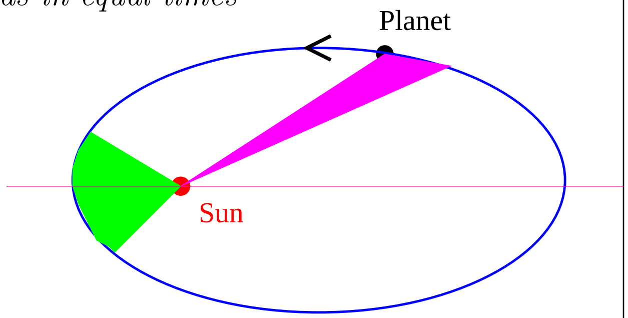
- *All planets move along elliptical orbits, with the Sun at one focus.*



- Most planetary orbits have small eccentricity. Comets have very eccentric orbits.
- The point of closest approach to the Sun is called the *perihelion* and the farthest point is the *aphelion*.

Kepler's Second Law

• *The line connecting the Sun to the planet sweeps out equal areas in equal times*



•This means that the planet has to move faster when it is nearer the Sun that when it is farther away: since the distance to the Sun is shorter, it has to sweep out a larger angle in the same time.

Conservation of Angular Momentum

•Let us turn this idea into an equation. Let the angle swept out by the radius vector in a time dt be $d\theta$. The area swept out is then $r^2 d\theta$. Kepler's second law says that $r^2 \frac{d\theta}{dt}$ is a constant. If we multiply this by the mass m of the planet we have

$$mr^2 \frac{d\theta}{dt} = \text{constant}.$$

•This says that the *angular momentum of the planet is a constant*. This would be true if the force acting on the planet is pointed along the line connecting the planet to the Sun: then this force would have no torque and cannot change the angular momentum of the planet.

Kepler's Third Law

• *The square of the period of a planet is proportional to the cube of the semi-major axis*

•This says that farther planets have smaller average angular velocity: they take longer to complete an orbit.

•We have, as an equation,

$$T^2 = Ca^3$$

for some constant C ; i.e., it is independent of the planet. So C is describing some property of the Sun.

Circular Orbits

•We can get a better feel for what Kepler's laws say if we ignore eccentricity for the moment. For circular orbits, the semi-major axis a is the same as the radius R . If the orbit is circular, the second law of Kepler says that the planet is in *uniform circular motion*: since the radius is constant, $\frac{d\theta}{dt}$ must itself be a constant.

•Now the acceleration of a particle in uniform circular motion is directed to the center. Moreover it has magnitude

$$R\omega^2 = R(\frac{2\pi}{T})^2 = (\frac{4\pi^2}{C})\frac{1}{R^2}.$$

•Thus the force acting on the planet is directed towards the Sun and has the magnitude

$$F = m(\frac{4\pi^2}{C})\frac{1}{R^2}$$

• *The force on the planet is proportional to its mass.*

• *The force on the planet is proportional to the inverse square of the distance.*

•We derived these facts assuming the orbits are all circular. The remarkable fact is that these two facts are sufficient to explain Kepler's laws for all cases, *even when the orbits are not circular.*

Newton's Universal Law of Gravitation

• *All bodies attract each other. The force of attraction is pointed along the line connecting the bodies and has magnitude*

$$\frac{Gm_1m_2}{R^2}$$

•Here, m_1, m_2 are the masses of the bodies and R the distance between them.

• G is a constant of nature which is the same for all objects. It is called *Newton's constant.*

•The quantity C we had in Kepler's laws now we see has to be related to the mass of the Sun:

$$(\frac{4\pi^2}{C}) = GM_{\text{Sun}}.$$

Thus the motions of planets allow

us to measure the combination GM_{Sun} .

•Newton's realization was that the same law explains the motion of projectiles on the Earth, the motion of the Moon around the Earth and planets around the Sun. For example, the acceleration due to gravity on the Earths surface is

$$g = G\frac{M_{\text{E}}}{R_{\text{E}}^2}$$

•Newton knew $g = 9.8\ ms^{-2}$, $R_{\text{E}} = 6.4 \times 10^6\ m$, so he could find GM_{E} :

$$GM_{\text{E}} = gR_{\text{E}}^2 = 4.01 \times 10^{14}\ \frac{m^3}{s^2}.$$

•Let us test the law of gravitation

by calculating the quantity GM_{E} another way: by using the orbit of the moon. Let us ignore the eccentricity of the moon's orbit.

•The distance to the Moon is $3.82 \times 10^8\ m$, its period is 27.3 days or $T = 2.36 \times 10^6\ s$. Now, we have applying Kepler's law to the Moon,

$$GM_{\text{E}} = 4\pi^2\frac{R^3}{T^2}$$

or

$$GM_{\text{E}} = 3.95 \times 10^{14}\ \frac{m^3}{s^2}$$

•It works!.

•It was not until over hundred years later that G was determined.