PHY 121 Mechanics *S.G. Rajeev*

Lecture 18

Newton's Law of Gravitation

Kepler's Laws for Circular Orbits

•We can get a better feel for what Kepler's laws say if we ignore eccentricity for the moment. For circular orbits, the semi-major axis a is the same as the radius R . If the orbit is circular, the second law of Kepler says that the planet is in *uniform circular motion*: since the radius is constant, $\frac{d\theta}{dt}$ must itself be a constant.

•Now the acceleration of a particle in uniform circular motion is directed to the center. Moreover it

has magnitude

$$R\omega^2 = R(\frac{2\pi}{T})^2 = (\frac{4\pi^2}{C})\frac{1}{R^2}.$$

•Thus the force acting on the planet is directed towards the Sun and has the magnitude

$$F = m(\frac{4\pi^2}{C})\frac{1}{R^2}$$

• *The force on the planet is proportional to its mass.*

• *The force on the planet is proportional to the inverse square of the distance.*

•We derived these facts assuming the orbits are all circular. The remarkable fact is that these two

facts are sufficient to explain Kepler's laws for all cases, *even when the orbits are not circular.*

Newton's Universal Law of Gravitation

• *All bodies attract each other. For point particles, the force of attraction is pointed along the line connecting the bodies and has magnitude*

$$\frac{Gm_1m_2}{R^2}$$

•Here, m_1, m_2 are the masses of the bodies and R the distance between them.

• G is a constant of nature which is the same for all objects. It is called *Newton's constant.*

•The quantity C we had in Kepler's laws now we see has to be related to the mass of the Sun:

$$(\frac{4\pi^2}{C}) = GM_{\rm Sun}.$$

Thus the motion of planets allow us to measure the combination GM_{Sun} .

•Newton's realization was that the same law explains the motion of projectiles on the Earth, the motion of the Moon around the Earth and planets around the Sun. For example, the acceleration due to gravity on the Earth's surface is

$$g = G\frac{M_{\text{E}}}{R_{\text{E}}^2}.$$

•Newton knew $g = 9.8\ ms^{-2}$, $R_{\text{E}} = 6.4 \times 10^6\ m$, so he could find GM_{E} :

$$GM_{\text{E}} = gR_{\text{E}}^2 = 4.01 \times 10^{14}\ \frac{m^3}{s^2}.$$

•Let us test the law of gravitation by calculating the quantity GM_{E} another way: by using the orbit of the Moon. Let us ignore the eccentricity of the Moon's orbit.

•The distance to the Moon is $3.82 \times 10^8\ m$, its period is 27.3 days or $T = 2.36 \times 10^6\ s$. Now, we have applying Kepler's law to the Moon,

$$GM_{\text{E}} = 4\pi^2\frac{R^3}{T^2}$$

or

$$GM_{\text{E}} = 3.95 \times 10^{14}\ \frac{m^3}{s^2}$$

•It works!.

The Shell Theorem

•But we made a leap in the above argument. The gravitational force was postulated to be $\frac{GMm}{R^2}$ only for point particles. How can we use this near the surface of a huge body like the Earth? It doesn't look like a point particle to me? A missing link is the 'shell theorem':

• *the force due to a* spherical *body is the same as that of a point particle of the same mass located at the center.*

• *Spherical* means here that the density of the body only depends on the distance from the center.

•It is enough to prove this theorem for an infinitesimally thin spherical shell of uniform density. Any spherical body can be thought of as a superposition of such shells and if the result is true for shells it will hold for any spherical body. That is why it is called the shell theorem.

•It was not for over hundred years later that G was determined to be $G = 6.67 \times 10^{-11}\ \ m^3/(kg\ s^2)$.

Gravitational Potential Energy

•Gravity is a *conservative force*. For we can find a function $U(\mathbf{r})$ such that the vector derivative is the force:

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$$

U is the gravitational potential energy .

•If you have a body of mass M at the origin, and another of mass m at $\mathbf{r}$, that potential energy is

$$U = -\frac{GMm}{|\mathbf{r}|}$$

•The force is $\mathbf{F} = -\frac{GMm}{r^2}\frac{\mathbf{r}}{r}$, where $r = |\mathbf{r}|$ is the distance between the bodies.

•To see this we have to differentiate $\frac{1}{r} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ with respect to x (for example) and show that is equal to $\frac{-x}{r^3}$.

•If we have many bodies around, their net force on m will be the vector sum of the individual gravitational forces. The total potential energy will be the sum

$$U(\mathbf{r}) = mG\frac{M_1}{r_1} + \frac{M_2}{r_2} + \cdots$$

•If there is a continuous distribution of matter of density $\rho(\mathbf{r})$, the gravitational potential energy of a mass m at the point $\mathbf{r}$ will be

$$U(\mathbf{r}) = -Gm\int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}dx'dy'dz'$$

Escape Speed

•Consider a body of mass m infinitely far removed at rest from the another of mass M. It has zero kinetic energy and potential eneergy. Now imagine that it falls in. At a distance r, it has a speed v say. Conservation of energy gives

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = 0$$

or,

$$v = \sqrt{(\frac{2GM}{r})}.$$

•Conversely this is the speed at which it must be shot out in order to barely escape to infinity. It is called the *escape speed* from that distance.

•On the Earth,

$$\frac{GM_E}{R_E^2} = g = 9.8\ \frac{m}{s^2}, \quad R_E = 6.37 \times 10^6\ m$$

so that

$$v_E = \sqrt{(2gR_E)} = 11.2\ \frac{km}{s}.$$

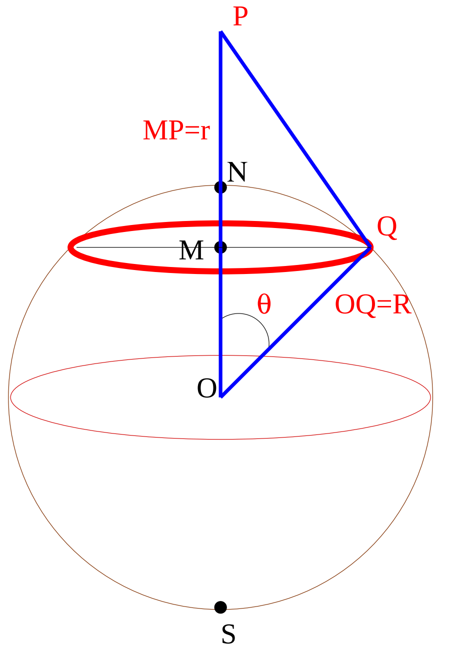
•In practice no object is launched at this velocity; the energy needed to boost a satellite to an orbit or to let it leave the Earth' gravitational field is given gradually by the rocket motors.

Appendix: Proof of the Shell Theorem

• *The gravitational potential energy due to a* spherical shell *of constant surface density is the same as that of a* point particle *of the same total mass as the shell located at its center.*

•**Proof** Refer to the diagram for the definition of the variables.

•Let σ be the surface density of the shell. Consider a circular ring between latitudes θ and $\theta + d\theta$. It has a radius if $R\sin\theta$ and a width of $Rd\theta$; hence an area of $2\pi R\sin\theta R d\theta$. So the mass of the ring is $2\pi\sigma R\sin\theta$.



•All the points on the ring are at the same distance $PQ = (R^2 +$

$r^2 - 2Rr\cos\theta)^{1/2}$ from the point $P = (x, y, z)$. The gravitational potential energy of a particle of mass m at a point $P = (x, y, z)$, due to the ring, is

$$dU = -Gm\frac{[2\pi\sigma R^2\sin\theta d\theta]}{(R^2 + r^2 - 2Rr\cos\theta)^{1/2}}.$$

•The angle θ varies from 0 at the N to π at the S . Thus the potential energy due to the whole shell is

$$U(r) = -Gm2\pi\sigma R^2 \int_0^\pi \frac{\sin\theta}{(R^2 + r^2 - 2Rr\cos\theta)^{1/2}}d\theta.$$

•This integral can be done by the

substitution $u = \cos\theta$:

$$\int_0^\pi \frac{\sin\theta}{(R^2 + r^2 - 2Rr\cos\theta)^{1/2}}d\theta = \int_{-1}^{1}[R^2 + r^2 - 2Rru]^{-\frac{1}{2}}du$$

which is

$$\frac{1}{2Rr\frac{1}{2}}[(R^2 + r^2 - 2Rru)^{\frac{1}{2}}]_{-1}^{1} = \frac{2}{r}.$$

•Thus

$$U(r) = -Gm2\pi\sigma R^2\frac{2}{r} = -\frac{Gm[4\pi R^2\sigma]}{r}$$

Since $4\pi R^2$ is the total area of the shell and $M = [4\pi R^2\sigma]$ its total mass, we have

$$U(r) = -\frac{GMm}{r}$$

•This is precisely the the potential energy due to a point of mass of total mass M at the center of the shell, as was to be proved.

•By differentiating this potential energy we can get that the force due to a spherical shell is also the same as that of a point particle at the center.