PHY 121 Mechanics *S.G. Rajeev*

Lecture 21

Oscillations

•Given enough time all systems tend to settle down to the state of lowest possible potential energy: this is a point of stable equilibrium.

•If it is moved away from this point, there will be a force that pushes it back to the equilibrium point. But often the system will overshoot and go over to the other side.

•It will then be pushed back again and overshoot to the other direction. Thus it will *oscillate* around the point of equilibrium.

•The most familiar example is the simple pendulum. We will study

this example later in greater detail.

•The time it takes for a system to start at one extreme position, and return to it is called the *period*. The number of complete oscilations per unit time is called the *frequency*.

•Clearly if T is the period, and f the frequency,

$$T = \frac{1}{f}.$$

Simple Harmonic Motion

•The most familiar periodic functions are $\sin$ and $\cos$. Thus perhaps the simplest kind of periodic motion would invlove them.

•A particle is performing *simple harmonic motion* if its position $x(t)$ at time t is given by

$$x(t) = A\cos(\omega t + \phi)$$

•The three constants A, ω, ϕ describe the motion completely. A is called the *amplitude* of the motion. It is the maximum possible value of $x(t)$, since the maximum possible value of $\cos$ is one.

•The constant ω is called *angular frequency*. The quantity $\omega t + \phi$ whose cosine we are taking can be thought of as an angle, and ω is the rate at which that angle is changing. *But this does not mean that this "angle" has any obvious geometrical meaning: we are not necessarily talking about a particle moving in a circle.*

•Recall that $\cos\theta$ is periodic with period 2π. Thus we have

$$\cos(\omega t + \phi + 2\pi) = \cos(\omega t + \phi).$$

On the other hand, T is the period of the motion so we must have

$$x(t) = x(t + T)$$

so that

$A\cos(\omega[t+T]+\phi) = A\cos(\omega t+\phi).$

•Comparing the two we get

$$\omega T = 2\pi$$

or

$$\omega = \frac{2\pi}{T} = 2\pi f.$$

•The constant ϕ is called the *phase*. If $\phi = 0$ the maximum displacement occurs at $t = 0$. Thus ϕ measures the deviation from the situation. For example if $\phi = \frac{\pi}{2}$ the particle is at the mean position at time $t = 0$: $x(0) = 0$. If $\phi = \pi$, $x(0) = -A$ which corresonds to maximum displacement but in the

negative direction.

Velocity and Acceleration of SHO

•If

$$x(t) = A\cos(\omega t+\phi)$$

we have the velocity

$$v(t) = \frac{dx(t)}{dt} = -A\omega\sin(\omega t+\phi).$$

•This is a periodic function of time also, with the same period.

•Also, the acceleration

$$a(t) = \frac{dv(t)}{dt} = -A\omega^2\cos(\omega t+\phi)$$

•We notice that in fact, for simple harmonic motion,

$$a(t) = -\omega^2 x(t)$$

•The force acting on the particle must be then

$$F = -m\omega^2 x.$$

For a particle executing Simple Harmonic Motion, the force is pointed towards the point of equilibrium; it has a magnitude proportional to the displacement

Hooke's Law for Springs

•Hooke's law is: *The force exerted by a spring is proportional to the displacement and is directed to the point of equilibrium.*

•If $F = -kx$, the constant k is called the spring constant or Hooke's constant.

•Hence a mass connected to a spring will execute simple harmonic motion. If $F = -kx$,

$$m\omega^2 = k$$

or,

$$\omega = \sqrt{(\frac{m}{k})}, \quad T = 2\pi\sqrt{(\frac{k}{m})}$$

Conservation of Energy

•A force proportional to the displacement is a conservative force. In fact if

$$F = -kx$$

the potential energy is

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2.$$

•We must have from conservation of energy,

$$\frac{1}{2}mv^2 + U(x) = \text{constant}.$$

•Let us see this explicitly. In fact

$$v = -A\omega\sin(\omega t + \phi).$$

The total energy is

$$\frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi) + \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \phi)$$

•But

$$\sin^2\theta + \cos^2\theta = 1$$

so that the total energy is

$$mA^2\omega^2$$

which is indeed a constant.

Uniform Circular Motion

•What if a particle is oscillating in two directions simultaneously? A simple example would be

$$x(t) = A\cos(\omega t), \quad y(t) = A\sin(\omega t).$$

•This is just uniform circular motion. Thus we can split up uniform circular motion into two pieces, each one being simple harmonic motion. Perhaps surprising since the circular motion looks at first very different from an oscillation.

•What if the amplitudes of oscillation in the two directions are not the same?

$$x(t) = A\cos(\omega t), \quad y(t) = B\sin(\omega t).$$

Then the motion is on an ellipse.

Prove this!

• In general very complicated shapes can arise by superposing simple harmonic motion of differing periods and amplitudes. In fact *any* periodic function can be written as a sum of cosines with various periods and amplitudes and phases. This is the procedure of *Fourier analysis*.

Damped Harmonic motion

• Most systems lose some energy to friction. So eventually the energy in the oscillation will decrease to zero: the amplitude will slowly decrease with time.

• The simplest way this can happen is for the amplitude to decrease exponentially with time:

$$x(t) = Ae^{-\gamma t}\cos(\omega t + \phi)$$

for some positive constant γ.

• It turns out that this corresponds to the case where the force is

$$F = -kx - 2m\gamma v.$$

It has a term proportional to velocity which is a sort of frictional force. It is thus not a conservative force.