PHY 121 Mechanics *S.G. Rajeev*

Lecture 3

Demonstration: Measurement of Acceleration

• *Galileo* made the first precise measurements on falling bodies. He discovered the *principle of equivalence*: in the absence of friction, *all bodies fall at the same acceleration due to gravity*. This is true not only for bodies falling straight down, but those falling down an inclined line, again if friction can be ignored.

•The air carts you see float on a cushion of air that is being pumped out of the holes in the track. So they move with almost no friction.

The Setup

•We will measure the acceleration of a few different bodies and see that it is independent of mass.

•We have two timers connected to the ancient Apple II computer here. As the cart moves through the track it interrupts a light beam, starting the first timer. The timer measures how long the beam is interrupted; i.e., how long it takes the cart to pass that point. Once we know the length of the cart this can be used to find the average velocity of the cart at that point. Similarly we measure the average velocity at the next timer.

The Theory Behind the Experiment

•Now suppose the length of the cart is L, the distance between the timers D and T_1, T_2 the times it takes for the cart to pass through each timer. How can we find the acceleration, a assuming that it is constant?

•Let v_1, v_2 be the (average) velocities at the two spots:

$$v_1 = \frac{L}{T_1}, \quad v_2 = \frac{L}{T_2}.$$

Let t be the time it takes for the cart to move from the first spot to the second.

•Then

$$v_2 = v_1 + at$$

and

$$D = v_1 t + \frac{1}{2}at^2.$$

•Solving the first equation for t we get

$$t = \frac{v_2 - v_1}{a}$$

and hence

$$D = \frac{v_1(v_2 - v_1)}{a} + \frac{1}{2}a\frac{(v_2 - v_1)^2}{a^2}$$

or,

$$D = \frac{(v_2 - v_1)(v_2 + v_1)}{2a} = \frac{v_2^2 - v_1^2}{2a}.$$

•Now we can solve for a :

$$a = \frac{v_2^2 - v_1^2}{2D}.$$

Finally we can write v_1, v_2 in terms of the times we measure:

$$a = \frac{L^2}{2D}(T_2^{-2} - T_1^{-2}).$$

Vectors

•A *Vector* is a quantity with a *magnitude* and a *direction*.

Examples:

• *Velocity* is a vector; its *magnitude* has a special name *speed*.

•Imagine a particle moving from an initial point ('origin') to a final point. A line connecting the two points represents its *displacement*, which is a vector.

•Imagine the particle first moved by a displacement $\mathbf{a}$ and then by $\mathbf{b}$. The net displacement $\mathbf{c}$ is the sum of these two vectors:

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

The vector $\mathbf{0}$ has length zero and satisfies $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for any $\mathbf{a}$.

Multiplication by Scalars

•A quantity that can be described by a real number is called a *scalar*.

•You can multiply vectors by real numbers. If x is a real number and $\mathbf{v}$ a vector, $x\ \mathbf{v}$ is a vector that has length $|x|$ times the length of $\mathbf{v}$. It points in the same direction as $\mathbf{v}$ if $x > 0$ and in the opposite direction if $x < 0$.

•The rules for addition and multiplication by a real number define the concept of a vector. It is useful to have the mental picture of a vector as an 'arrow'.

Components

•We usually use a right-handed co–ordinate system to describe the co–ordinates of any point. There are three special vectors of length one associated to a co–ordinate system: $\mathbf{i}$ pointing along the first (or x axis), $\mathbf{j}$ along the second (or y axis) and $\mathbf{k}$ along the third (or z axis.) These are also called sometimes $\hat{x}, \hat{y}, \hat{z}$ respectively.

•A point that happens to be along the x -axis can be represented the vector $x\mathbf{i}$. More generally, if its co–ordinates are (x, y, z) it is represented by the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The numbers (x, y, z) are called the *components* of this vector.

Addition by Components

•If their sum

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

has components

$$c_1 = a_1 + b_1, \quad c_2 = a_2 + b_2, \quad c_3 = a_3 + b_3.$$

The Scalar Product

•We saw that there is a way to multiply a *vector* by a *scalar* to get a *vector*.

•Now we will see how to multiply two *vectors* to get a *scalar*. This is operation is called a *scalar product* beacuse the end result is a scalar. It is also called a *dot product* since we write it as $\mathbf{a} \cdot \mathbf{b}$.

•The *scalar product* of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \phi$$

where ϕ is the angle between the two vectors.

•The scalar product is independent of the order of multiplication:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

If we interchange $\mathbf{a}$ and $\mathbf{b}$, the angle ϕ between them will become $-\phi$ but since $\cos(\phi) = \cos(-\phi)$ the product remains unchanged.

•The scalar product of a vector with itself is equal to the square of its length.

•The scalar product of two orthogonal vectors is zero.

•If we multiply any of the vectors by a number, the scalar product also gets multiplied by the same number:

$$(\alpha \mathbf{a}) \cdot \mathbf{b} = \alpha \mathbf{a} \cdot \mathbf{b}$$

Scalar Product in Terms of Components

•The *scalar product* satisfies the following basic property:

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$$

•This leads to another formula for scalar product in terms of components. If

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

and

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

This is a very useful formula.

Example
Let

$\mathbf{a} = 9\mathbf{i} - 5\mathbf{k}, \quad \mathbf{b} = -2\mathbf{i} + 13\mathbf{k}.$

What is $\mathbf{a} \cdot \mathbf{b}$? What is the angle between them?
Solution
We have two ways of writing the dot product:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

where θ is the angle between the two vectors. After a couple of steps,

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

Moreover,

$$|\mathbf{a}| = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$$

etc. Thus we have

$\mathbf{a} \cdot \mathbf{b} = 9 \times (-2) + (-5) \times (13) = -83.$

Also,

$|\mathbf{a}| = \sqrt{(a_1^2 + a_2^2 + a_3^2)} = \sqrt{(9^2 + 5^2)} = 10.30$

$|\mathbf{b}| = \sqrt{(b_1^2 + b_2^2 + b_3^2)} = \sqrt{(2^2 + 13^2)} = 13.15.$

Thus

$\theta = \arccos\left(\frac{-83}{10.30 \times 13.15}\right) = 127.8^\circ$

The Vector Product

•There is also a way to multiply two vectors to get a new vector. This operation is denoted by $\mathbf{a} \times \mathbf{b}$ and is hence also called the 'cross product'.

•The Vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is of magnitude $|\mathbf{a}||\mathbf{b}||\sin\theta|$, θ again being the angle between them. The direction of $\mathbf{a} \times \mathbf{b}$ is perpendicular to *both* $\mathbf{a}$ and $\mathbf{b}$. If you curl the fingers of your *right* hand along the smaller angle connecting $\mathbf{a}$ to $\mathbf{b}$, your thumb will point in the direction of the cross product.

Vector Product in Component Form

•A more convenient rule is in terms of components. If

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

and

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

the components of $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ are given by

$$c_1 = a_2 b_3 - a_3 b_2$$

$$c_2 = a_3 b_1 - a_1 b_3$$

$$c_3 = a_1 b_2 - a_2 b_1.$$

Some Facts about Vector Products

•In particular we have the following facts that you can verify:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

•In the vector product, *the order of multiplication matters.*

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

Vector multiplication is not commutative.

•In particular

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

since $\mathbf{a} \times \mathbf{a} = -\mathbf{a} \times \mathbf{a}$.

•Any two parallel vectors in fact will have zero cross product. Thus $\mathbf{a} \times \mathbf{b} = 0$ need *NOT* mean that

one of the two factors is zero.