PHY 121 Mechanics *S.G. Rajeev*

PHY 121 Lecture 4

Projectile Motion

•The position of a particle at any instant can be thought of as a vector, usually called $\mathbf{r}$ with components (x, y, z) :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

•The motion of a particle can be described by a function that assigns to any real number t, representing time, a vector $\mathbf{r}(t)$

•The *velocity* at time t is the derivative of position:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

Acceleration

•The *acceleration* at time t is the derivative of velocity:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2}$$

•Since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are constants, we can get simple formulas for the velocity and acceleration in terms of the components of the position vectors:

$$\mathbf{v}(t) = \frac{dx(t)}{dt}\mathbf{i} + \frac{dy(t)}{dt}\mathbf{j} + \frac{dz(t)}{dt}\mathbf{k}$$

$$\mathbf{a}(t) = \frac{d^2x(t)}{dt^2}\mathbf{i} + \frac{d^2y(t)}{dt^2}\mathbf{j} + \frac{d^2z(t)}{dt^2}\mathbf{k}$$

Motion of a Projectile in the Vertical Direction

•If you throw a stone straight up, its motion has constant acceleration due to gravity. Suppose we choose a co–ordinate system with the y-axis pointing upwards. The x, z co-ordinates are contants and

$$y(t) = y_0 + u_y t - \frac{1}{2}gt^2$$

where $g = 9.8\ ms^{-2}$ is the magnitude of the acceleration due to gravity. The minus sign indicates that the acceleration in fact points downward. y_0 is the y co-ordinate of position at time t and u_y the initial velocity.

Projectile Motion in the General Case.

•Now suppose we throw the stone at some angle, not straight up. Then it turns out that the acceleration is still pointing straight down. The sideways component of the velocity is constant.

•Thus we can split up the motion of the particle into three separate functions, one for each co-ordinate.

$$x(t) = x_0 + u_x t$$

$$y(t) = y_0 + u_y t - \frac{1}{2}gt^2$$

and finally

$$z(t) = z_0 + u_z t$$

Example

A cannon is fired at an angle θ to the horizontal and a muzzle speed u. What is the maximum height H attained?

Solution

We have the initial velocity

$$\mathbf{u} = u\cos\theta\mathbf{i} + u\sin\theta\mathbf{j}$$

and constant acceleration $-g\mathbf{j}$. Hence the motion along the horizontal direction is given by (we choose the gun to the origin of the co–ordinate system)

$$x(t) = u\cos(\theta)t$$

while

$$y(t) = u\sin(\theta)t - \frac{1}{2}gt^2.$$

The maximum height is just the maximum value of y. Let T be the time at which the maximum height is attained. Using the fact that at a maximum the derivative is zero,

$$u\sin\theta - gT = 0$$

so that

$$T = \frac{u\sin\theta}{g}$$

and

$$H = \frac{1}{2}\frac{u^2\sin^2\theta}{g}.$$

A projectile moves along a parabola

•Using the notation of the above example, the vertical co–ordinate increases from zero to a maximum value of H and then drops back to zero. We can find a relation between the x and y co–ordinates by eliminating t:

$$t = \frac{x(t)}{u\cos\theta}$$

and

$$y = x\frac{\sin\theta}{\cos\theta} - \frac{1}{2}\frac{g}{u^2\cos^2\theta}x^2.$$

This is the equation for a parabola. (That a projectile follows a path in the shape of a parabola was known even to Archemedes.)

The Range of a Cannon

•Over flat terrain, the *range* of the projectile will be the distance it travels when y returns to 0. From the above, the range D is

$$D = \frac{u^2}{g}2\sin\theta\cos\theta = \frac{u^2}{g}\sin 2\theta$$

•Thus to maximize the range with a fixed muzzle velocity of the cannon, we must have $\theta = 45°$.

•In the 'real world' there are two additional facts to consider: air friction decreases the true range below the above value. Also for long enough distances we can no longer treat the earth as flat.