PHY 121 Mechanics *S.G. Rajeev*

Lecture 5

Uniform Circular Motion

•Suppose a particle is moving according to the equation

$$\mathbf{r}(t) = R\cos\omega t\mathbf{i} + R\sin\omega t\mathbf{j}$$

where R, ω are constants. What is its velocity and acceleration?

•The particle is moving in a circle, since the orbit lies in a plane and the distance to the origin, $|\mathbf{r}(t)|$ is constant. The velocity at the instant t is obtained by differentiating each component,

$$\mathbf{v}(t) = -R\omega\sin(\omega t)\mathbf{i} + R\omega\cos(\omega t)\mathbf{j}$$

Direction of Velocity and Acceleration

•The *velocity* is *tangential* to the circle; i.e., *perpendicular to the radius*. Indeed:

$$\begin{aligned}\mathbf{v}(t)\cdot\mathbf{r}(t) &= [-R^2\omega\sin(\omega t)\cos(\omega t) \\ &\quad +R^2\omega\sin(\omega t)\cos(\omega t)] \\ &= 0.\end{aligned}$$

•By differentiating the velocity,

$$\mathbf{a}(t) = -R\omega^2\cos(\omega t)\mathbf{i} - R\omega^2\sin(\omega t)\mathbf{j}$$

Thus

$$\mathbf{a}(t) = -\omega^2\mathbf{r}(t).$$

The acceleration points towards the center.

Magnitude of Velocity and Acceleration

•The acceleration and velocity are both of constant magnitude:

$$v = |\mathbf{v}(t)| = R\omega$$

$$a = |\mathbf{a}(t)| = R\omega^2.$$

Indeed we can eliminate the constant ω to get

$$a = \frac{v^2}{R}.$$

Thus in uniform circular motion acceleration is pointed directly towards the center of the circle with the above magnitude.

Angular Velocity

•The quantity ω is called *angular velocity*. It describes the rate at which the angle that the position vector of the particle makes with a fixed line is changing.

•If T is the period of revolution, we have

$$2\pi R = vT$$

since $2\pi R$ is the circumference of a circle of radius R .

Isaac Newton's Flight of Fancy

•Instead of a real experiment we will perform a 'thought experiment' today. If you think carefully enough, with just the right mix of imagination and rigor you can understand much about physics. But this can be dangerous in the wrong hands: there are countless crackpots with too much imagination who have derived nonsense this way.

• *Don't try thought experiments at home.* The chances of you getting a correct answer are small.

•Imagine you are on the surface of a planet with no atmosphere (so we can ignore air friction.) You have a cannon whose muzzle velocity you can control at will. You shoot a ball; it lands at a point not far from you. You increase the muzzle velocity, it falls further away.

•You keep increasing the muzzle velocity and the ball falls further and further away. Indeed the distance is now becoming comparable to the radius of the planet.

•As you cranck up the muzzle velocity even further, the balls 'falls' beyond your horizon. Eventually for a sufficiently great muzzle velocity it will fall at the point opposite to you on the surface of the planet.

•As you increase the velocity even further the ball starts to make almost a complete revolution around the planet and falls not too far from you but *in the opposite direction*.

•You keep increasing the muzzle velocity. The ball now makes one complete revolution and whizzes past you *without touching the ground*.

•It will *forever* orbit the planet while *falling* freely.

You have just turned the ball into a satellite.

•Newton showed by this argument that the laws that govern celestial bodies like the Moon are no different from those that govern bodies on the Earth. The Moon orbits the Earth and the Earth orbits the Sun by the same laws that govern the motion of cannon balls.

•The Moon is falling freely towards the Earth; its acceleration vector is pointed at the Earth.

Motion of a Satellite

- The acceleration due to gravity at the distance at which most artificial satellites orbit the Earth is almost the same as that at Earth's surface: $9.8\ ms^{-2}$. What is their orbital period?
- We have from previous discussion of uniform circular motion,

$$T = \frac{2\pi r}{v}$$

and the magnitude of acceleration

$$a = \frac{v^2}{r}$$

so that

$$v = \sqrt{(ra)}$$

and

$$T = 2\pi\sqrt{(\frac{r}{a})}$$

- If we are orbiting just above the Earth's atmosphere, the value of r is almost the radius of the Earth and that of a is almost the acceleration due to gravity at the Earth's surface.

$$r = 6.37 \times 10^6\ m, \quad a = 9.8\ ms^{-2}$$

so that

$$T = 5.066 \times 10^3\ s$$

or about one and a half hours.

- *Our analysis is valid only if the distance to the satellite is small compared to the radius of the Earth. It does not apply to the Moon.*