John Golden

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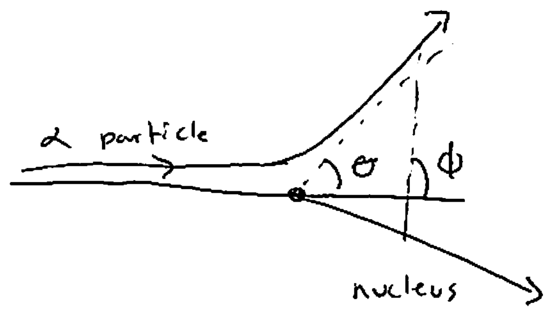
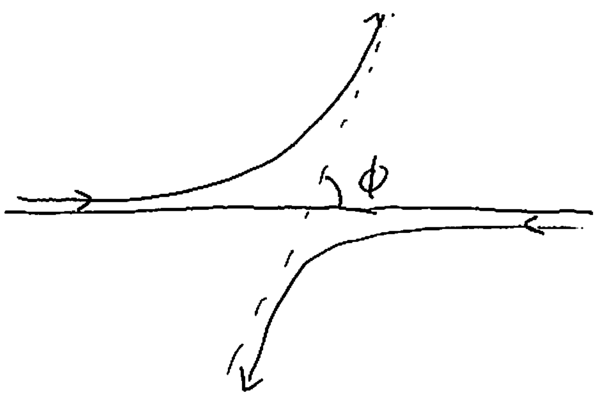
Problem Set 1
Solutions

1)



← Lab frame



← Center of Mass frame

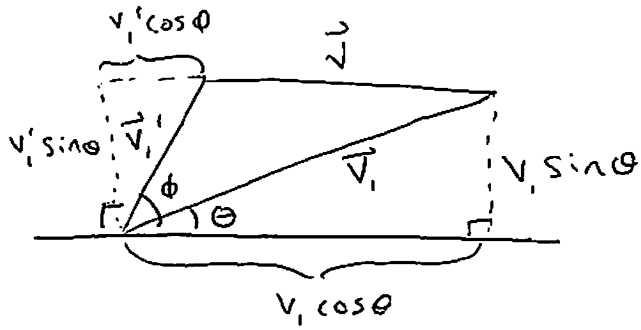
Define

$\vec{r}_1$, $\vec{v}_1$ as post-impact position + velocity of α-particle in Lab frame

$\vec{r}_1'$, $\vec{v}_1'$ Same as above but in CoM frame

$\vec{R}$, $\vec{V}$ position + velocity of center of mass in Lab frame

$$\Rightarrow \vec{r}_1 = \vec{R} + \vec{r}_1' \quad \Rightarrow \vec{v}_1 = \vec{V} + \vec{v}_1'$$

$\Rightarrow$ 

By definition,

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\Rightarrow \quad (m_1 + m_2)\vec{R} = m_1 \vec{r}_1 + m_2 \vec{r}_2$$

$$\Rightarrow \quad (m_1 + m_2)\vec{V} = m_1 \vec{v}_1 + m_2 \vec{V}_2$$

Conservation of momentum gives

$$m_1 \vec{v}_1 + m_2 \vec{V}_2 = m_1 \vec{V}_0$$ (where $\vec{V}_0$ is the α-particle's original speed)

$$\Rightarrow \quad \vec{V}(m_1 + m_2) = m_1 \vec{V}_0$$

$$\Rightarrow \quad \vec{V} = \frac{\mu}{m_2} \vec{V}_0$$ (with reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$)

From the diagram

$$V_1 \sin\theta = v_1' \sin\phi \quad , \quad V_1 \cos\theta = v_1' \cos\phi + V = v_1' \cos\phi + \frac{\mu}{m_2} V_0$$

$$\Rightarrow \quad \frac{V_1 \sin\theta}{V_1 \cos\theta} = \tan\theta = v_1' \frac{\sin\phi}{v_1' \cos\phi + \frac{\mu}{m_2} V_0}$$

$$\Rightarrow \tan\theta = \frac{\sin\varphi}{\cos\varphi + \frac{\mu}{m_2}\frac{V_0}{V_1'}}$$

Recall

post-impact velocity of nucleus

$$V_1 = V + V_1' \Rightarrow V_1' = V_1 - V = V_1 - \frac{m_1V_1 + m_2V_2}{m_1 + m_2}$$

$$= \frac{(m_1V_1 + m_2V_1) - (m_1V_1 + m_2V_2)}{m_1 + m_2}$$

$$= \frac{m_2V_1 - m_2V_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}(V_1 - V_2) = \frac{\mu}{m_1}(V_1 - V_2) = V_1'$$

Collision is elastic $\Rightarrow$ KE is conserved $\Rightarrow$ relative velocity of the particles stays the same $\Rightarrow V_0 = V_1 - V_2$

$$\Rightarrow V_1' = \frac{\mu}{m_1}V_0 \Rightarrow \frac{\cancel{\mu}}{m_2}\frac{V_0}{V_1'} = \frac{\cancel{\mu}}{m_2}\cancel{V_0}\cdot\frac{m_1}{\cancel{\mu}\cancel{V_0}} = \frac{m_1}{m_2}$$

$$\Rightarrow \boxed{\tan\theta = \frac{\sin\varphi}{\cos\varphi + \frac{m_1}{m_2}}}$$

2. As is the standard method in calculus, we take the derivative of our previously obtained $\Theta[\phi] = \tan^{-1}\left(\frac{\sin\phi}{\cos\phi + \frac{m_1}{m_2}}\right)$ with respect to ϕ, set that equal to zero to find the location of extremum, then plug that back in to the original function to get the value at the extremum.

$$\Rightarrow \frac{d}{d\varphi}\Theta[\varphi] = \frac{(\cos\varphi + \frac{m_1}{m_2})\cos\varphi - \sin\varphi(-\sin\varphi)}{(\frac{m_1}{m_2} + \cos\varphi)^2}\left(\frac{1}{\frac{\sin^2\varphi}{(\frac{m_1}{m_2} + \cos\varphi)^2} + 1}\right)$$

$$= \frac{(\cos\varphi + \frac{m_1}{m_2})\cos\varphi + \sin^2\varphi}{\sin^2\varphi + (\frac{m_1}{m_2} + \cos\varphi)^2} = 0$$

$$\Rightarrow (\cos\varphi + \frac{m_1}{m_2})\cos\varphi = -\sin^2\varphi$$

$$\Rightarrow \cos^2\varphi + \sin^2\varphi + \frac{m_1}{m_2}\cos\varphi = 0$$

$$\Rightarrow \frac{m_1}{m_2}\cos\varphi = -1 \Rightarrow \varphi_{max} = \cos^{-1}\left(\frac{-m_2}{m_1}\right)$$

$$\Rightarrow \Theta(\varphi_{max}) = \tan^{-1}\left(\frac{\sin(\cos^{-1}(\frac{-m_2}{m_1}))}{-\frac{m_2}{m_1} + \frac{m_1}{m_2}}\right)$$

$$\boxed{\Theta_{max} = \tan^{-1}\left(\frac{\sqrt{1+(\frac{m_2}{m})^2}}{-\frac{m_2}{m_1} + \frac{m_1}{m_2}}\right)}, \quad t \equiv \frac{m_2}{m_1}$$

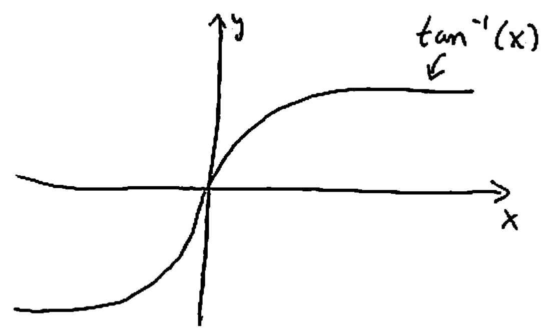
This is the final answer w/out approximations

$$\Rightarrow \theta_{max} = \tan^{-1}\left(\frac{\sqrt{1+\epsilon^2}}{\frac{1}{\epsilon} - \epsilon}\right)$$

Because ϵ is very small we neglect all ϵ terms of order 2 or higher. This is standard approximation.

$$\Rightarrow \theta_{max} = \tan^{-1}\left(\frac{1}{\frac{1-\epsilon^2}{\epsilon}}\right)$$

$$\Rightarrow \theta_{max} = \tan^{-1}(\epsilon)$$



Clearly as $\epsilon \to 0$, $\tan^{-1}(\epsilon) \to 0$

So $\boxed{\theta_{max} \text{ is small}}$

3. All this is asking for is how much energy the alpha particle needs to "hit" the Au nucleus, i.e., assuming the α-particle starts at infinity, how much energy it will need to get to within 8.15 fm of a charge equivalent to a gold nucleus ($79e$). This is just the potential at that distance, so

$$\text{Energy} = \frac{1}{4\pi\epsilon_0}\frac{(2e)(79e)}{8.15\text{ fm}} = \frac{158\,(1.602\times10^{-19})^2}{4\pi(8.85\times10^{-12})(8.15\times10^{-15})}\ \text{J}$$

$$= 4.457 \times 10^{-12}\ J$$

$$1\ J = 6.24 \times 10^{18}\ eV$$

$$\Rightarrow 4.457 \times 10^{-12} \times (6.24 \times 10^{18}) = 2.79 \times 10^{7}\ eV$$

$$\Rightarrow \boxed{27.9\ MeV}$$

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Problem Set 2
Solutions

John Colden

4.1

\# of Nuclei decayed $= N_o - N(t)$, and $dN(t) = -\alpha N(t)\,dt \Rightarrow N(t) = N_o e^{-\alpha t}$

$$\frac{N(T_{1/2})}{N(0)} = \frac{1}{2} = e^{-\alpha T_{1/2}} \Rightarrow \ln 2 = \alpha T_{1/2} \Rightarrow N(t) = N_o e^{-\ln 2 \frac{t}{T_{1/2}}}$$

$\Rightarrow$ \# of Nuclei decayed $= D(t) = N_o\left(1 - e^{-\ln 2 \frac{t}{T_{1/2}}}\right)$, now $t \ll T_{1/2} \Rightarrow$ let $\frac{t}{T_{1/2}} \equiv \epsilon$

Taylor expansion of $e^{-\ln 2\, \epsilon} = 1 - \ln 2\, \epsilon + O(\epsilon^2)$ as ϵ is small

$$\Rightarrow D(t) = N_o(1 - 1 + \ln 2\, \epsilon) \Rightarrow \boxed{D(t) = N_o \ln 2 \frac{t}{T_{1/2}}}$$

4.2

$D(t) = \lambda \Rightarrow$ prob. of each nucleus to decay is $\frac{\lambda}{N_o} = p$

$$\Rightarrow P(r) = \binom{N_o}{r} p^r (1-p)^{N_o - r} \approx \frac{N_o^r}{r!} p^r (1-p)^{\overbrace{N_o - r}^{\approx N_o}} = \frac{N_o^r}{r!}\left(\frac{\lambda}{N_o}\right)^r \left(1 - \frac{\lambda}{N_o}\right)^{N_o}$$

$$\approx \frac{\lambda^r}{r!} e^{-\lambda}, \quad \text{as } \left(1 - \frac{\lambda}{N}\right)^N \to e^{-\lambda} \text{ for large } N$$

$$\Rightarrow \boxed{P(r) = \frac{\lambda^r}{r!} e^{-\lambda}}$$

5.1

$$N_0 = (10^{-3}\ \text{kg})\left(\frac{1\ \text{amu}}{1.66\times10^{-27}\ \text{kg}}\right)\left(\frac{1\ \text{atom}}{266\ \text{amu}}\right) = \boxed{2.67\times10^{21}\ \text{atoms}}$$

5.2

$$\lambda = N_0\left(1 - e^{-\ln 2\,\frac{t}{T_{1/2}}}\right) \Rightarrow \left(1 - \frac{\lambda}{N_0}\right) = e^{-\ln 2\,\frac{t}{T_{1/2}}} \Rightarrow T = \frac{-\ln 2\, t}{\ln\left(1 - \frac{\lambda}{N_0}\right)}$$

We have $\lambda = 3.7\times10^{10}$ decays, $N_0 = 2.67\times10^{21}$ atoms, $t = 1\ s$, plugging in gets

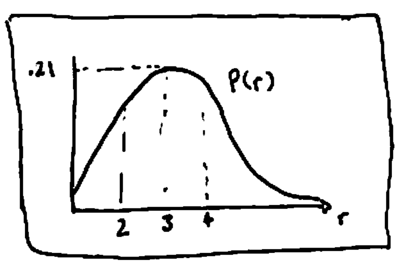
$$\Rightarrow T = 5\times10^{10}\ s \Rightarrow \boxed{T = 1600\ \text{yrs}}$$

5.3

For $t \ll T_{1/2}$, our decay rate is going to be constant. Therefore our average # of decays per 0.1 s per 10^{-9} g will be

$$\lambda = 1\ \text{Curie}\cdot 10^{-10} = 3.7$$

$$\Rightarrow \boxed{P(r) = \frac{3.7^r}{r!}e^{-3.7}}$$



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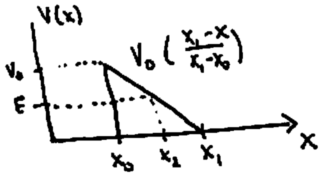

As given in the notes, using the WKB approximation, the probability of tunneling is given by

$$P = \exp\left[-\frac{2}{\hbar}\sqrt{2m}\int_{x_0}^{x_2}\sqrt{V - E}\,dx\right]$$

6 cont.

where $V(x_2) = E$.

$$\int_{x_0}^{x_2} \sqrt{V-E}\, dx = \sqrt{E} \int_{x_0}^{x_2} \sqrt{\frac{V}{E} - 1}\, dx$$, let $u = \frac{V}{E} = \frac{V_0}{E} \frac{x_1 - x}{x_1 - x_0}$

$\Rightarrow du = \frac{-V_0}{E(x_1 - x_0)} dx$

$$= -\sqrt{E}\, \frac{E(x_1 - x_0)}{V_0} \int_{V_0/E}^{1} \sqrt{u-1}\, du$$

$u(x_0) = \frac{V_0}{E}$, $u(x_2) = \frac{V(x_2)}{E} = 1$

$$= -\sqrt{E}\, \frac{E(x_1 - x_0)}{V_0} \frac{2}{3} (u-1)^{3/2} \Big|_{V_0/E}^{1} = \frac{2}{3} \frac{x_1 - x_0}{V_0} (V_0 - E)^{3/2}$$

$$\Rightarrow \boxed{P = \exp\left[\frac{-4\sqrt{2m}}{3\hbar V_0} (x_1 - x_0)(V_0 - E)^{3/2} \right]}$$

where x_2 is given by $V(x_2) = E \Rightarrow V_0 \dfrac{x_1 - x_2}{x_1 - x_0} = E \Rightarrow x_2 = x_1 - \dfrac{E}{V_0}(x_1 - x_0)$

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Problem Set 3
Solutions

7.1 $[L_i, L_j] = [r_j p_k - r_k p_j, r_k p_i - r_i p_k]$

$= [r_j p_k, r_k p_i] + [r_k p_j, r_i p_k] - [r_j p_k, r_i p_k] - [r_k p_j, r_k p_i]$

$[r_\alpha, p_\beta] = i\hbar \delta_{\alpha\beta}$

$\Rightarrow r_j [p_k, r_k] p_i + r_i [r_k, p_k] p_j$

$= -i\hbar r_j p_i + i\hbar r_i p_j = i\hbar (r_i p_j - r_j p_i) = i\hbar L_k$

$\Longrightarrow \boxed{[L_i, L_j] = i\hbar \epsilon_{ijk} L_k}$

7.2

$[L^2, L_i] = [\cancel{L_i^2}, L_i] + [L_j^2, L_i] + [L_k^2, L_i]$

$= L_j [L_j, L_i] + [L_j, L_i] L_j + L_k [L_k, L_i] + [L_k, L_i] L_k$

$= -i\hbar L_j L_k - i\hbar L_k L_j + L_k (i\hbar L_j) + i\hbar L_j L_k$

$= i\hbar (-L_j L_k + L_k L_j - L_k L_j + L_j L_k) = 0$

$\Longrightarrow \boxed{[L^2, L_i] = 0 \text{ for arbitrary } i}$

7.3

The eigenvalues of L^2 are $\hbar^2 \ell(\ell+1)$, where ℓ is an $\frac{\text{integer}}{2} \geq 0$.

The eigenvalues of L_3 are $m\hbar$, where m is an integer with $m \leq |\ell|$.

7.4

If we let $L_\pm \equiv L_1 \pm i L_2$, then we have

$L^2 = L_+ L_- - \hbar L_3 + L_3^2$, and it is easily shown that

$[L_3, L_\pm] = \pm \hbar L_\pm$, so $L_3 L_\pm |\ell, m\rangle = (L_\pm L_3 \pm \hbar L_\pm)|\ell, m\rangle$

$= L_\pm L_3 |\ell, m\rangle \pm \hbar L_\pm |\ell, m\rangle = m\hbar L_\pm |\ell, m\rangle \pm \hbar L_\pm |\ell, m\rangle = (m \pm 1)\hbar L_\pm |\ell, m\rangle$,

That is, $L_\pm$ raises or lowers the value of m by one unit.

Therefore we see that

$$L_\pm |\ell, m\rangle = \sqrt{(\ell \mp m)(\ell \pm m + 1)}\ \hbar\ |\ell, m \pm 1\rangle$$

So $L_\pm = L_1 \pm i L_2$

$$\Rightarrow (L_1)_{ij} = \frac{\hbar}{2}\left[\sqrt{(\ell - m)(\ell + m + 1)}\ \delta_{i,j+1} + \sqrt{(\ell + m)(\ell - m + 1)}\ \delta_{i,j-1}\right]$$

$$(L_2)_{ij} = \frac{i\hbar}{2}\left[\quad " \quad - \quad " \quad\right]$$

And naturally $(L_3)_{ij} = j\hbar\,\delta_{ij}$ and $(L^2)_{ij} = \ell(\ell+1)\hbar^2\,\delta_{ij}$

So with $m = j$ we can construct our matrices:

$\underline{\ell = 1}$ - 3x3 matrices

$$L^2 = 2\hbar^2\,\mathbb{1}_3, \quad L_3 = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}, \quad L_1 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad L_2 = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}$$

$\underline{\ell = 2}$ - 5x5 matrices

$$L^2 = 6\hbar^2\,\mathbb{1}_5, \quad L_3 = \hbar\begin{pmatrix}2&&&&\\&1&&&\\&&0&&\\&&&-1&\\&&&&-2\end{pmatrix}, \quad L_1 = \frac{\hbar}{2}\begin{pmatrix}0&2&&&\\2&0&\sqrt{6}&&\\&\sqrt{6}&0&\sqrt{6}&\\&&\sqrt{6}&0&2\\&&&2&0\end{pmatrix}, \quad L_2 = \frac{i\hbar}{2}\begin{pmatrix}0&-2&&&\\2&0&-\sqrt{6}&&\\&\sqrt{6}&0&-\sqrt{6}&\\&&\sqrt{6}&0&-2\\&&&2&0\end{pmatrix}$$

$\underline{\ell = 3}$ - 7x7 matrices

$$L^2 = 12\hbar^2\,\mathbb{1}_7 \qquad L_3 = \hbar\begin{pmatrix}3&&&&&&\\&2&&&&&\\&&1&&&&\\&&&0&&&\\&&&&-1&&\\&&&&&-2&\\&&&&&&-3\end{pmatrix}, \quad L_1 = \frac{\hbar}{2}\begin{pmatrix}0&\sqrt{6}&&&&&\\\sqrt{6}&0&\sqrt{10}&&&&\\&\sqrt{10}&0&\sqrt{12}&&&\\&&\sqrt{12}&0&\sqrt{12}&&\\&&&\sqrt{12}&0&\sqrt{10}&\\&&&&\sqrt{10}&0&\sqrt{6}\\&&&&&\sqrt{6}&0\end{pmatrix}$$

$$L_2 = \frac{i\hbar}{2}\begin{pmatrix}0&-\sqrt{6}&&&&&\\\sqrt{6}&0&-\sqrt{10}&&&&\\&\sqrt{10}&0&-\sqrt{12}&&&\\&&\sqrt{12}&0&-\sqrt{12}&&\\&&&\sqrt{12}&0&-\sqrt{10}&\\&&&&\sqrt{10}&0&-\sqrt{6}\\&&&&&\sqrt{6}&0\end{pmatrix}$$

(and $\ell = 0 \Rightarrow L^2 = L_1 = L_2 = L_3 = 0$)

8. Denote spin-up state by $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and spin-down state by $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We wish to find S_x, S_y, S_z. From prob. 7,

We know $S_z|+\rangle = \frac{\hbar}{2}|+\rangle$, $S_z|-\rangle = -\frac{\hbar}{2}|-\rangle$

$$\Rightarrow S_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right) = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We also know that $S_\pm \equiv S_x \pm i S_y$ satisfies

$S_+|-\rangle = \hbar|+\rangle$, $S_-|+\rangle = \hbar|-\rangle$, and $S_+|+\rangle = S_-|-\rangle = 0$

$$\Rightarrow S_+ = \hbar|+\rangle\langle-| = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S_- = \hbar|-\rangle\langle+| = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$\Rightarrow \boxed{S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

These are the Pauli spin matrices.

8.1 The possible combinations of states for two particles are (denoted $|m_1, m_2\rangle$)

$$\left.\begin{array}{l} |+,+\rangle \\ \frac{1}{\sqrt{2}}\left(|+,-\rangle + |-,+\rangle\right) \\ |-,-\rangle \end{array}\right\} \text{Spin } 1$$

and $\frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right)$ → spin 0

8.1 cont. So clearly the available spins are

$\boxed{1, 0}$

8.2

Under the change $m_1 \rightarrow m_2$, it is clear that the three Spin-1 combinations undergo no change, that is, they are Symmetric. The spin-zero case would change sign under this transformation, and is thus anti-symmetric.

$$\Rightarrow \boxed{\begin{array}{l} \text{Spin } 1 \Rightarrow \text{symmetric} \\ \text{Spin } 0 \Rightarrow \text{anti-symmetric} \end{array}}$$

9. $$H = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}$$

$$I_1 = I_2 \Rightarrow \quad H = \frac{L_1^2 + L_2^2}{2I_1} + \frac{L_3^2}{2I_3}, \quad L_1^2 + L_2^2 = L^2 - L_3^2$$

$$\Rightarrow H = \frac{L^2 - L_3^2}{2I_1} + \frac{L_3^2}{2I_3} = \frac{1}{2}\left(\frac{(L^2 - L_3^2) I_3 - I_1 L_3^2}{I_1 I_3} \right)$$

$$\Rightarrow H = \frac{1}{2}\left(\frac{L^2}{I_1} + L_3^2 \left(\frac{I_1 - I_3}{I_1 I_3} \right) \right)$$

So $H|l,m\rangle = \boxed{\frac{\hbar^2}{2I_1} l(l+1) + \frac{\hbar^2 m^2}{2}\left(\frac{I_1 - I_3}{I_1 I_3}\right)}$

9.2. $\underline{I_1 = I_2 > I_3}$

$\Rightarrow$ $I_1 - I_3$ is positive, so for a fixed l, picking $m=0$ will minimize E

$$\Rightarrow \boxed{E_{min} = \frac{\hbar^2}{2I_1} l(l+1)}$$

$\underline{I_1 = I_2 < I_3}$

$\Rightarrow$ $I_1 - I_3$ is negative, so taking $m = \pm l$ will lower the energy more than setting $m=0$, so

$$E_{min} = \frac{\hbar^2}{2I_1} l(l+1) + \frac{\hbar^2 l^2}{2}\left(\frac{1}{I_3} - \frac{1}{I_1}\right)$$

$$\Rightarrow \boxed{E_{min} = \frac{\hbar^2}{2}\left(\frac{l}{I_1} + \frac{l^2}{I_3}\right)}$$