Complex Analysis and Differential Equations, PHY 401/OPT411 U. of Rochester Fall 2001

S. G. Rajeev

February 6, 2002

Contents

Syllabus

1. The field of complex numbers; norm of a complex number;the complex plane; stereographic map of the sphere.
2. Closed and open sets; convergent sequences; infinite series; absolute convergence; double series. Infinite products.
3. Definition of a continuous function of a complex variable;Definition of an analytic function; Cauchy–Riemann equations; harmonic functions.
4. Polynomials; Power series; exponential and related functions.
5. Contour integrals; Cauchy's theorem; Taylor's theorem; Laurent's theorem. Residue calculus; Principal value; Hilbert transform.
7. Entire functions; Product representation of trigonometric functions. Weierstrass functions. The Gamma function. Product formula. Asymptotic formula.
8.(Time Permitting) Divergent series. Borel summation; Pade approximants.
1. Linear ordinary differential equations; ordinary point,regular and irregular singular points; the point at infinity; Local behavior near regular singular points.
2. Solution near an ordinary point; Wronskian; solution near a regular singular point;
3. The hyper-geometric equation. Solution by power series. Integral representation. Behavior at infinity.
4.Bessel functions; Legendre functions; Hermite's equation. Sturm–Liouville problems.
5. (Time Permitting) Local behavior near irregular singular points; asymptotic series.
6. Fourier and Laplace transforms; solution of differential equations by Fourier and Laplace transforms.
7. Linear second order partial differential equations; canonical form; hyperbolic, elliptic and parabolic types; Cauchy problem for the wave equation.

Chapter 1

Complex Numbers

1 Mathematics originated in the construction of altars for religious ceremonies in pre-historic times.

1.1 Among the first discoveries where properties of triangles and methods for construction of various geometric shapes using a rope and a stick. Later, the architecture of temples, the study of the heavens, the development of calendars, the need for standard weights and measures, keeping of financial accounts, classification of musical scales, grammatical formulation of languages, all led to mathematical developments.

2 We begin with the natural numbers $1, 2, 3\cdots$.

2.1 They form an *ordered set*: for every pair of numbers a, b, either $a < b$ or $b < a$ or $a = b$. Thus natural numbers form an increasing sequence starting with the smallest, 1, then its successor 2, and its successor 3 and so on. This is the basis of counting, the most basic of all mathematical operations.

3 Natural numbers can be added and multiplied to get others.

3.1 The product of any number with 1 is itself. By analogy we introduce a new number *zero* 0, which when added to any number gives itself.

4 To solve linear equations, enlarge the set of numbers to the *rational numbers*.

4.1 Linear equations are of the form $cx + b = a$, where x is the unknown. When a, b, c are integers, sometimes this equation has a natural number as solution; e.g., $3x + 1 = 7 \Rightarrow x = 2$ .But in general there is no natural number solution: $6x + 4 = 1$. To solve these equations we must introduce the notion of rational numbers; in the above example $x = -\frac{1}{2}$. The word *rational* comes from *ratio*. Even when the parameters a, b, c are rational, the linear equation $ax + b = c$ has a rational solution.

5 Rational numbers form an *ordered field.*

5.1 A *field* is any set on which the operations of addition, multliplication and division (except by *zero*) are defined. These must satisfy the axioms

1. $a + b = b + a$; addition is *commutative*

2. $a + (b + c) = (a + b) + c$; addition is *associative*

3. $a + 0 = a$; 0 is the *additive identity*

4. For every rational number a there is another, $-a$ such that $a+(-a) = 0$; existence of an additive *inverse*

5. $ab = ba$; multiplication is commutative

6. $a(bc) = (ab)c$; multiplication is associative

7. $1a = a$; 1 is the multiplicative identity

8. For every $a \neq 0$ there is another, $\frac{1}{a}$ such that $a(\frac{1}{a}) = 1$; existence of a multiplicative *inverse*

9. $a(b + c) = ab + ac$;multiplication is *distributive* over addition

6 There are lengths in geometry not measurable by rational numbers.

6.1 Numbers can be thought of as measuring lengths of straight line segments in some units. But not all line segments constructed by simple geometric methods correspond to rational lengths: the hypotenuse of a right angle triangle with whose base and altitude are both 1 is not a rational number.

6.2 Think of a straight line extending in both directions without limit. Choose a point on it, the *origin*. We can regard positive rational numbers as the distances (in some standard unit) of points to the right of the origin and negative numbers as representing points to the left. It is now easy to see that there are points not represented by rational numbers- there would be many 'holes' in the line if we only allowed rational distances. And yet, in between any two rational numbers there would be another rational number; e.g., $\frac{a+b}{2}$ is in between a and b . Thus these holes are of vanishingly small size. Any point on the line can be approximated with arbitrary accuracy by rational number- as the accuracy increases, we need larger and larger denominators or more and more decimal places. Thus we can think of points on the line as represented by such convergent sequences of rational numbers. This is a very intricate construction. See Chapter 11 *Introduction to Metric and Topological Spaces* by Sutherland, for details.

7 *Real numbers* correspond to points on a straight line.

7.1 For now, we will accept the notion of real numbers as given: the intuition based on thinking of the real numbers as points on the real line will be enough.

8 They form a *complete* ordered field.

8.1 We already know about an ordered field: the rational numbers form one. The new notion is that of completeness. To make it precise we need the notion of convergence, which we postpone to the next chapter. But we give here the basic notion of metric and norm.

8.2 There is a *norm* on the set of real numbers: the magnitude $|x|$ of a real number is equal to itself if it is positive and otherwise is $-a$. It satisfies $|ab| = |a||b|$.

8.3 The set of real numbers is a *metric space*. That is, there is a notion of distance or *metric* on the space of real numbers: $d(a,b) = |b-a|$. This is simply the distance between the points representing the numbers on the real line. The distance satisfies the *triangle inequality* $d(a,c) \leq d(a,b) + d(b,c)$ for any three numbers a, b, c . Moreover $d(a,b) = 0$ if and only if $a = b$.

9 There are quadratic equations that cannot solved by real numbers.

9.1 The most obvious one is $z^2 + 1 = 0$. This forces us to expand our notion of numbers even further.

10 A *complex number* in an ordered pair of real numbers $z = (x, y)$.

10.1 We define addition and multiplication of complex numbers as follows:

$$(a, b) + (x, y) = (a + x, b + y), \quad (a, b)(x, y) = (ax - by, ay + bx).$$

These satisfy all the familiar properties. That is,

11 The complex numbers form a field.

11.1 We see that $(x, y)(1, 0) = (x, y)$ so that we can identify $(1, 0) = 1$. Moreover, $(0, 1)^2 = -1$. This number is special and is denoted by $i = (0, 1)$. Thus any complex number can be written uniquely as $(x, y) = x + iy$, where x and y are real numbers. For historical reasons the first component x is called the *real part* and the second the *imaginary part*.

11.2 Complex numbers can be visualized as points on the plane. The magnitude of a complex number is its distance from the origin: $|z| = \sqrt{[x^2 + y^2]}$; its argument is its angle in the polar co-ordinate system: $\text{Arg } z = \arctan \frac{y}{x}$. If $|z| = r, \text{Arg } z = \theta$, $z = re^{i\theta}$.

11.3 The set of complex numbers is also a metric space. The distance between two complex numbers is just the distance between the points on the plane representing them: $d(z.z') = |z - z'|$. It obviously satisfies the triangle inequality.

12 The set of complex numbers is algebraically closed.

12.1 An algebraic equation of order n is $a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0 = 0$ with $a_n \neq 0$. We know that even if the coefficients $a_0, a_1, \cdots a_n$ are real, the real solutions for z do not always exist. But there is always a complex number that solves this equation: we dont need to expand the notion of number any further. We will not prove this statement-known as the *Fundamental Theorem of Algebra*- here.

13 Quantum mechanics is based on the notion of a complex vector space.

13.1 Complex numbers are merely useful mathematical gadgets in classical physics. However, the very basic axioms of quantum mechanics involve complex numbers. Hence they are as 'physical' as as any of the other numbers we described above.

13.2 There are other number systems which are algebraically complete and even complete with respect to a different notion of metric-the *p -adic numbers*. But they dont appear in physics so we don't discuss them in this course.

14 By including the point at infinity, complex numbers describe the points on a sphere.

14.1 Imagine a sphere that touches the plane at exactly one point. Call this point the South pole. From its antipode (North Pole) draw a straightline to another point on the sphere. If this line is extended it will meet a point on the plane.

14.2 This establishes a one-one correspondence (*stereographic projection*) between the points on the sphere (minus the North pole) and the points on the sphere. As we approach the north pole, the point corresponding to it on the plane moves away to infinity. In fact it is as if the North pole corresponds to the (unique) 'point at infinity'. We will see that this geometric picture is useful in analysis.

Chapter 2

Convergence

15 The notion of a convergent sequence is central to analysis.

15.1 Let us begin by attempting to solve the equation $x^2 = 2$. There is clearly no solution in terms of rational numbers. Yet we can find a sequence of rational numbers that approach the solution arbitrarily closely. It is clear that the answer lies between one and two. Let us change variables to $x = 1+y$ so that y is smaller than one. Then the equation can be rewritten as $y = \frac{1-y^2}{2}$. Since y is smaller than one, y^2 will be even smaller so we can think of the nonlinear term on the rhs as a 'small correction'. Thus $x_1 = \frac{3}{2}$ is our next approximation.

15.2 This method can now be iterated, giving the following recursion relation $y_{n+1} = \frac{1-y_n^2}{2}$ and the sequence of approximations $x_n = 1 + y_n$:

$$\begin{aligned}\sqrt{2} &= \{1, \frac{3}{2}, \frac{11}{8}, \frac{183}{128}, \frac{46127}{32768}, \frac{3042762591}{2147483648} \cdots\} \\ &= \{1., 1.5, 1.375, 1.42969, 1.40768, 1.4169, 1.4131, 1.41467, 1.41402, 1.41429, 1.41418, \\ &\quad 1.41423, 1.41421, 1.41422, 1.41421, 1.41421, \cdots\}\end{aligned}$$

We see that these numbers are getting closer and closer so that the interval between two successive terms becomes smaller and smaller. It is then intuitively obvious that this sequence tends to a limit: then this limit will be a solution of our equation.

15.3 Many problems of analysis are solved in this way by successive approximations. We can apply them to functions rather than numbers.

16 A *sequence* is a function from the set of natural numbers to any set

16.1 The n th element of the sequence s is usually written as s_n .

17 A *metric* on any set X is a function $d: X \times X \to R$ such that

$$d(x,y) = d(y,x), \quad d(x,y) \geq 0, \quad d(x,y) = 0 \iff x = y, \quad d(x,y) \leq d(x,z) + d(z,y)$$

A set along with a metric is called a *metric space*.

17.1 The obvious notion of distance between complex numbers

$$|z - z'| = \surd[(x - x')^2 + y - y')^2], \quad z = x + iy, \quad z' = x' + iy'$$

satisfies these conditions. There are other notions of metric on the set of complex numbers as well. For example we can regard the complex numbers are points on a sphere (instead of the plane) and take the distance between points along great circles that connect them. We will return to this idea later.

18 A sequence s_n on a metric space X *converges* to a point $x \in X$ if, for any $\epsilon > 0$, there is an N such that $d(s_n, x) < \epsilon$ for all $n > N$. We will write $\lim_{n\to\infty} s_n = x$ or $s_n \to x$ for short.

18.1 In other words the distance between the points in the sequence and the point x can be made as small as we want by choosing n big enough.

19 A sequence can converge at most to one point.

20 A sequence of complex numbers is *bounded* if there exists a number B (the *bound*) such that $|s_n| \leq B$.

20.1 There are many bounded sequences which are not convergent. For example think of a sequence the even terms of which are approximations to $\surd 2$ while the odd terms are approximations to $\surd 3$. This doesn't converge to either point.

21 A subset S of the real line is *bounded above* if there is a number B such that all elements of S are less than or equal to B. The smallest such upper bound is called the *supremum* or $\sup$ for short.

21.1 The notion of *infimum* or $\inf$ is defined analogously for sequences that are bounded below, as the greatest lower bound.

22 A bounded sequence of real numbers will have a well-defined $\sup$ and $\inf$ even if it is not convergent.

22.1 For example the sequence $x_{2n} = 2, x_{2n+1} = 3$, is not convergent and has $\sup x_n = 3, \inf x_n = 2$.

23 z is a *limit point* of a sequence a_n if every circle centered at z contains an infinite number of points of the sequence.

23.1 A convergent sequence will have only a single limit point. But a bounded sequence may have many limit points. Indeed it can even have an infinite number of them.

24 The *limit supremum* ($\limsup$) of a real sequence that is bounded above is the supremum of its limit points. Similarly we define $\liminf$.

24.1 What is the difference between $\sup_n a_n$ and $\limsup_n a_n$ of a sequence of real numbers?. The $\limsup$ is sensitive only to the limit points, so a finite number of large entries will not affect its value; but the supremum will depend on these as well.

24.2 For example ,consider the sequence

$$x_n = 0, 1, 2, 3, 1.0, 1.5, 1.375, 1.42969, 1.40768, 1.4169, 1.4131, 1.41467, 1.41402, 1.41429, 1.41418$$

which is a finite sequence $0, 1, 2, 3,$ followed by the successive approximations to $\sqrt{2}$. The $\sup$ of this sequence is 3 while the $\limsup$ is $\sqrt{2}$.

25 A sequence s_n is a *Cauchy sequence* if for every ϵ there is an N such that $d(s_n, s_m) < \epsilon$ for every $m, n > N$.

25.1 What this means is that we can make the distance between two terms as small as we wish by moving far enough down the sequence.

26 Every Cauchy sequence of complex numbers has a limit. A metric space on which every Cauchy sequence has a limit within itself is *complete*.

26.1 The set of rational numbers is not a complete metric space. Show that the sequence of approximations to $\sqrt{2}$ defined above is a Cauchy sequence. It doesn't have a rational limit.

26.2 Which of the following are Cauchy sequences?

$$s_n = \log n, \quad s_n = \sum_{k=1}^{n} \frac{1}{k^2}, \quad s_n = \sum_{k=1}^{n} \frac{1}{k}$$

27 If the partial sums $S_n = \sum_{k=1}^{n} s_n$ form a convergent sequence, we say that the sum is *convergent*. We denote $\lim_{n\to\infty} \sum_{k=1}^{n} s_n = \sum_{k=1}^{\infty} s_k$.

27.1 Show that $\sum_1^\infty \frac{1}{n!}$ is convergent. What is the most rapidly converging series (with rational terms) for π that you can come up with?

28 A series $\sum_{k=1}^{\infty} a_k$ is *absolutely convergent* if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

28.1 Absolute convergence is a stronger condition than convergence; a series may converge because of cancellations among terms, but the sum of the magnitudes may not converge; e.g., $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ is convergent but not absolutely convergent.

28.2 We are allowed to rearrange terms of an absolutely convergent series; in general that is not allowed: the cancellations required to make a series converge may be lost if it is rearranged. We are not allowed, in the above example, to sum all the even terms then all the odd terms.

29 A series $\sum_n a_n$ converges absolutely if $\lim\sup |a_n|^{\frac{1}{n}} < 1$; it diverges if $\lim\sup |a_n|^{\frac{1}{n}} > 1$.

29.1 This is known as the *Cauchy test*. The point is that if $\limsup |a_n|^{\frac{1}{n}} < 1$, for all except a finite number of values of n , $|a_n| < R^n$ for some $R < 1$. Then the series converges by comparison with the geometric series.

29.2 Similarly if $R > 1$, the series will diverge.

29.3 The borderline cases where $\limsup |a_n|^{\frac{1}{n}} = 1$ require much more sophisticated analysis to decide convergence; in fact there is as yet no sure-fire test of convergence for an arbitrary sequence.

29.4 The series $\sum_{n=1}^{\infty} \frac{z^n}{n^k}$ converges when $|z| < 1$ and diverges when $|z| > 1$ since, $\limsup_n \left|\frac{z^n}{n^k}\right|^{\frac{1}{n}} = |z| \limsup e^{-\frac{k}{n}\log n} = |z|$.

30 If all except a finite number of elements of the sequence $1 + a_n$ are non-zero, and the partial products $P_N = \prod_{k=1}^{N}[1 + a_n]$ form a convergent sequence, we define $\prod_{k=1}^{\infty}[1 + a_n] = \lim_{N\to\infty} P_N$.

31 The infinite product $\prod_{k=1}^{\infty}[1+a_n]$ converges if the series $\sum_n |a_n|$ converges

31.1 This is based on the fact that if $\sum |a_n|$ converges, $\sum |a_n|^k$ converges as well for any $k > 1$. We can bound the product by such series. See *Whittaker and Watson* for proof.

31.2 We regard a product with an infinite number of zero factors as divergent although in some sense its value is just zero. The point is that it is a kind of 'essential zero', and behaves in a singular manner.

32 A function $f : R \to R$ is *continuous* at $x_0 \in R$ if for all $\epsilon > 0$, there is a δ such that $|f(x) - f(x_0)| < \epsilon$, for all x with $|x - x_0| < \delta$.

32.1 In other words a continuous function can be made to change as little needed by changing its argument by a small enough amount.

32.2 The function (*sign function*), $\text{sgn}\ (x) = +1$ for $x > 0$, $\text{sgn}\ (0) = 0$, $\text{sgn}\ (x) = -1$ for $x < 0$ is continuous everywhere except at the origin. The function which is equal to one for rational numbers and equal to 0 for irrational numbers is continuous nowhere.

32.3 Is $\sin\frac{1}{x}$ a continuous function at $x = 0$?What about $x \sin\frac{1}{x}$?

33 The space of bounded continuous functions $f : [a, b] \to C$ is itself a metric space, with the distance

$$d(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

33.1 In fact this is a complete metric space. This now allows us to think of convergent sequences of functions: all Cauchy sequences of continuous functions tend to a limit that is also continuous .

33.2 The distance between a pair of functions is the largest difference between their values at a common point.

34 We can now define a convergent sequence of functions using the above metric.

34.1 In more detail, we say that a sequence of functions $f_n : [a, b] \to R$ is *uniformly convergent* to a function f if, for every $\epsilon > 0$, there is an N_ϵ (that may depend on ϵ but not on x) such that

$$|f_n(x) - f(x)| < \epsilon,$$

for all $n > N_\epsilon$.

34.2 This is stronger than mere convergence of the sequence of real numbers $f_n(x)$ to the number $f(x)$ at each x : the rate of convergence must also be the same for all x , so that N_ϵ is independent of x . That is the meaning of the word *uniform*.

34.3 Uniform convergence is the same as the convergence in the $\sup$ norm defined earlier; for, if the *largest* difference between the values is less than ϵ that is the same as saying that the difference at all points is less than ϵ .

35 The space of continuous functions is a complete metric space under the $\sup$ norm.

35.1 Thus Cauchy sequences of continuous functions in this norm will converge to continuous functions. The set of polynomials form a dense subset: any continuous function can be approximated as close as we want by polynomials. There are other classes of dense sets; e.g., Fourier series.

35.2 This allows us to make sense of infinite sums of continous functions.

Chapter 3

Power Series

36 A *power series* is a series of the form $\sum_{r=0}^{\infty} a_r z^r$, where a_r are a sequence of complex numbers independent of z .

36.1 We might occasionally consider power series centered at a point other than the origin, $\sum_{r=0}^{\infty} a_r (z - z_0)^r$. Series with negative powers will not be called power series; they are called *Laurent series* instead. We will study them later.

37 The Cauchy test (comparison with geometric series) shows that a power series converges if $\limsup_n |a_n|^{\frac{1}{n}} |z| < 1$.

37.1 That is, it converges in the circle of radius $R = [\limsup_n |a_n|^{\frac{1}{n}}]^{-1}$. This is the *radius of convergence*.

37.2 Of course there are power series that have zero radius of convergence; e.g., $\sum_{n=0}^{\infty} n! z^n$. They are still useful in physics, but we will need more sophisticated methods to study them. We will return to them later.

37.3 The standard of comparison is the geometric series $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ which converges in the unit circle. The series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has infinite radius of convergence: it converges for any z .

38 A function $f : C \to C$ is *analytic at the point* z_0 if it is equal to some power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ within the circle of convergence.

38.1 The series must have non-zero radius of convergence for this to make sense.

38.2 This idea of an analytic function is the single most important concept in complex analysis. Therefore it is useful to have several different points of view on it. Later on we will see that a function is analytic if and only if it satisfies a partial differential equation (Cauchy-Riemann equation); or a certain integral equation (Cauchy's integral formula.) Thus we will find three equivalent criteria for analyticity. If we choose one as the definition (as above), the others can be derived as theorems.

38.3 The definition we give in terms of power series is more elementary than that found in many textbooks such as the one of *Copson*. It is also the one that generalizes easily to the case of several complex variable and to algebraic geometry. See for example the classic text of *Shafarevitch*.

39 A series with an infinite radius of convergence defines an *entire function*.

39.1 Polynomials are entire functions; the exponential is another example.

40 The sin and cos functions are defined by the infinite series of Madhava[1]

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

[1]Madhava (1340-1425) is the founder of a school of mathematics that flourished for about two hundred years. This school (Parameswara, Jyeshtadeva, Neelakanta Somayaji, Achyutha Pisharody...) laid the foundations of real analysis: the notion of convergence, infinite series, expansions for trigonometric functions, evaluation of area by quadrature, term by term differentiation of infinite power series, iterative solution of nonlinear equations are all due to them. Most of these results are attributed often to Newton. In the process of solving trigonometric equations by iteration Madhava even discovered some chaotic phenomena. Madhava also discovered the series

$$x = \tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x - \cdots$$

called often the Gregory series. Ironically, Madhava is most famous for the addition formula for sin , a result not original to him. Complex analysis is primarily due to the European school of mathematicians starting with Euler (Cauchy, Weierstrass, Riemann ...).

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

40.1 Prove addition formula for $\sin$ using the binomial expansion:

$$\sin[x+y] = \sin x \cos y + \cos x \sin y.$$

The periodicity of the trigonometric functions is not obvious from their series expansions. How would you establish periodicity?

41 The exponential function is defined by the power series

$$\exp(z) = \sum_{r=0}^{\infty} \frac{z^r}{r!}.$$

42 $\sin, \cos, \exp$ **are examples of entire functions.**

42.1 Using $n! = n^n \prod_{k=1}^{n-1} [1 - \frac{k}{n}]$ we get $\log[n!]^{\frac{1}{n}} = \log n + \frac{1}{n} \sum_{k=1}^{n-1} \log[1 - \frac{k}{n}]$. In the limit of large n the Euler-Maclaurin approximation formula for integrals gives a finite answer for the last sum:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \log[1 - \frac{k}{n}] = \int_0^1 dx \log[1-x] dx = -1.$$

Thus $[n!]^{\frac{1}{n}} = \mathrm{O}(n)$. That is why the sum for the exponential has infinite radius of convergence.

42.2 The exponential function satisfies the identity

$$\exp(z)\exp(z') = \exp(z+z')$$

which can be proved by resumming the double series using the binomial theorem

$$\sum_{n=0}^{r} \frac{r!}{n![r-n]!} z^n z'^{r-n} = (z+z')^r.$$

The rearrangement of terms in this power series is justified because the series converges uniformly. Hence

$$\exp(z) = e^z$$

where

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

43 Now it is clear that

$$e^{ix} = \cos x + i \sin x$$

43.1 Prove this by comparing the series expansions. In particular we have the formual

$$e^{i\pi} + 1 = 0.$$

44 Interesting power series can be obtained by differentiation or integration of familar series.

44.1 For example,

$$[1 - z]^{-2} = \sum_{n=1}^{\infty} n z^{n-1}, \quad [1 - z]^{-k} = \sum_{0}^{\infty} \frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!} z^n.$$

These are special cases of the binomial theorem.

45 The logarithm is defined by the series

$$-\log[1 - z] = z + \frac{z^2}{2} + \frac{z^3}{3} \cdots$$

45.1 Differentiation (and change of variables $z \to 1 - z$) gives

$$\frac{d}{dz}\log z = \frac{1}{z}$$

46 The exponential of an entire function is also entire.

46.1 Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ have radius of convergence R . We note first that (by collecting terms with the same power of z),

$$f^2(z) = \sum_{n_1=0}\sum_{n_2=0} f_{n_1} f_{n_2} z^{n_1+n_2} = \sum_{N=0}^{\infty} z^N \sum_{n_1+n_2=N} f_{n_1} f_{n_2}.$$

More generally,

$$f^r(z) = \sum_{N=0}^{\infty} z^N \sum_{n_1+n_2\cdots n_r=N} f_{n_1} f_{n_2}\cdots f_{n_r}.$$

Now summing both sides after multiplying by $\frac{1}{r!}$,

$$e^{f(z)} = \sum_{N=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{n_1+n_2\cdots n_r=N} f_{n_1} f_{n_2}\cdots f_{n_r}.$$

Now, since the series $\sum_{n=0}^{\infty} f_n z^n$ is convergent with radius of convergence R , $\limsup_n |f_n|^{\frac{1}{n}} = R$. Thus the coefficients $|f_n|$ grow at most as R^{-n} as $n \to \infty$. Then $|f_{n_1}\cdots f_{n_r}|$ grows at most like R^{-N} where $N = n_1 + n_2 \cdots n_r$ .Thus

$$|\sum_{n_1+n_2\cdots n_r=N} f_{n_1}\cdots f_{n_r}| = O(P_r(N)R^{-N})$$

where $P_r(N)$ is the number of solutions to the equation $n_1 + \cdots n_r = N$. This is at most $(N+1)^r$ since each part has to be less than or equal to the whole. Thus we can estimate the sum over r to give

$$|\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{n_1+n_2\cdots n_r=N} f_{n_1} f_{n_2}\cdots f_{n_r}| = O(R^{-N}e^{N+1}).$$

So the sum over N itself can thus be bounded by a geometric series with radius of convergence $e^{-1}R$.

46.2 If $f(z)$ is entire, its series converges for any R ; then the series for $e^{f(z)}$ converges for any radius as well.

46.3 We can show by expanding around the point $z=1$ that $e^{\log z}=z$, at least within a circle of radius one around 1 .

47 A series with negative powers of z , $\sum_{n=-N} a_n z^n$ is a *Laurent Series* if the power series contained in it, $\sum_{n=0}^{\infty} a_n z^n$ has a non-zero radius of convergence.

47.1 Thus a Laurent series is a polynomial in $\frac{1}{z}$ plus a power series in z with positive radius of convergence.

47.2 The simplest examples are $\frac{1}{z}, \frac{1}{z^2}+z$ etc.

47.3 If there is only one negative power of z (i.e., $N=1$) we say that the function has a *simple pole* at the point $z=0$. If $a_N \neq 0$ and $a_n=0$ for $n<N$, we say that the function has a pole of order N . The coefficeint of $\frac{1}{z}$, (i.e., a_{-1}) is called the *residue*.

47.4 Thus $\frac{2}{z}+\frac{1}{z^3}$ has a pole of order 3 at the origin, with a residue equal to 2 .

47.5 We can expand around a point other than the origin by using a change of variable; $\sum_{n=-N}^{\infty} a_n (z-z_0)^n$.

48 The function $\frac{1}{1-z}$ has a convergent power series expansion around any point $z_0 \neq 1$.

48.1 Thus it is analytic everywhere except at 1 , where it has a simple pole of residue -1 .

48.2 For $z_0 \neq 0$,

$$\frac{1}{1-z}=\frac{1}{1-z_0}\frac{1}{1-\frac{z-z_0}{1-z_0}}=\frac{1}{1-z_0}\sum_{n=0}^{\infty}\frac{1}{(1-z_0)^n}(z-z_0)^n.$$

This series converges around a circle centered at z_0 with radius $|1-z_0|$.

48.3 When k is a positive integer, we see that $(1-z)^{-k}$ is analytic everywhere except for a pole of order k at $z=1$.

48.4 Now consider the series

$$g(z) = 1 + \frac{1}{2}z + \frac{\frac{1}{2}[\frac{1}{2}-1]}{2!}z^2 + \frac{\frac{1}{2}[\frac{1}{2}-1][\frac{1}{2}-2]}{3!}z^3 + \cdots$$

This series also converges in the circle $|z| < 1$. We can expect that it is equal to $(1+z)^{\frac{1}{2}}$ by comparison with the binomial theorem for integer powers. It is possible to show by collecting together terms with the same power of z that $g^2(z) = 1+z$ confirming this suspicion. However,the square root is not a single-vlaued function: there are two solutions to the equation $g^2(z) = 1+z$ .The solution we pick is the one that assigns positive roots to positive real numbers: the *principal branch*. This rule will break down as we aproache the negative real axis: there is a *branch cut* along the negative real axis where the square root changes sign.

48.5 Another example of such a function, with an infinite number of branches is the logarithm. Since $e^{2\pi i} = 1$, the logarithm is ambiguous by an integer multiple of $2\pi i$. Our definition

$$\log z = \log[1-(1-z)] = [z-1] - \frac{[z-1]^2}{2} + \frac{[z-1]^3}{3} + \cdots$$

resolves this ambiguity by pricking the *principal branch*: the logarithm of a positive real numver is real. But along the negative real axis in z is a branch cut; crossing this line will make the logarithm jump by a multiple of $2\pi i$. We could have placed the cut along any line connecting the origin to infinity, but by convention we place it along the negative real axis.

Chapter 4

Cauchy-Riemann Equations

49 A function $f : C \to C$ may be thought of as a function of two real variables, since a complex number is just a pair of real numbers: $z = (x, y) = x + iy$.

49.1 If it is differentiable as a function of these real variables, we can define the operators $\partial, \bar{\partial}$ by

$$\partial f = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right], \quad \bar{\partial} f = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right].$$

49.2 It is as if we differentiate with respect to z and $\bar{z}$ rather than x, y :

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \partial f dz + \bar{\partial} f d\bar{z}$$

where $dz = dx + idy, d\bar{z} = dx - idy$.

49.3 We can check easily that $\partial z = 1, \bar{\partial} z = 0$ and more generally that, $\partial z^n = nz^{n-1}, \bar{\partial} z^n = 0$ for a natural number $n = 1, 2, 3 \cdots$. Also, $\bar{\partial} \bar{z}^n = n\bar{z}^{n-1}, \partial \bar{z}^n = 0$. Thus we can pretend that z and $\bar{z}$ are independent variables and that $\partial, \bar{\partial}$ are partial derivatives with respect to them.

50 An analytic function satisfies the Cauchy-Riemann equations:

$$\bar{\partial} f = 0$$

50.1 An analytic function is a power series in z alone without involving $\bar{z}$. The Cauchy-Riemann equations simply assert this independence on $\bar{z}$.

50.2 Thus the function $z\bar{z} = x^2 + y^2$ is not analytic: it invloves $\bar{z}$. It does not satisfy the Cauchy-Riemann equations either.

50.3 An archaic point of view (found for example in *Copson*) is that the Cauchy-Riemann equations describe the independence of the derivative on the direction in which $dz \to 0$. This confuses analyticity with differentiability, so we will not use it here.Also it does not generalize to the case of functions of several complex variables. However the Cauchy-Riemann equations do have a natural generalization: that df is a differential form of type $(1, 0)$.

50.4 The Cauchy-Riemann equation is an example of a partial differential equation. We will see that it can be solved in terms of the boundary value on any closed curve. This is known as the Cauchy integral formula.

50.5 An anti-analytic function is a power series (with non-zero radius of convergence) in $\bar{z}$ alone: it is the complex conjugate of an analytic function. It satisfies $\partial f = 0$.

50.6 $\Delta = \partial\bar{\partial}$ is known as the *Laplace operator*; $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. A function satisfying *Laplace's equation* $\Delta f = 0$ in a region Ω is said to be *harmonic* in Ω .

51 A harmonic function is the sum of an analytic and an anti-analytic function.

51.1 We will return to the topic of partial differential equations later. For now we will be content with the theory of the Cauchy-Riemann equation.

Chapter 5

Cauchy's Integral Theorems

52 A *curve* on the complex plane is a continuous function $\gamma : [a, b] \to C$.

52.1 It is *piece-wise smooth* if there are sub-intervals in each of which the function has continuous derivatives of arbitrary order.

53 The integral of a function $f : C \to C$ along a smooth curve $\gamma : [a, b] \to R$ is

$$\int_\gamma f dz = \int_a^b f(\gamma(t)) \frac{d\gamma}{dt} dt$$

53.1 On a piece-wise smooth curve it is defined as the sum over each of the segments on which it is smooth.

53.2 This is a special case of a more general notion of the *line integral* of a *differential 1-form*. $A = A_x dx + A_y dy$ is a differential 1-form on the plane where A_x, A_y are a pair of complex valued smooth functions. Its integral along a curve $\gamma : [a, b] \to R^2$ is

$$\int_\gamma A = \int_a^b [A_x(\gamma(t)) \frac{d\gamma_x(t)}{dt} + A_y(\gamma(t)) \frac{d\gamma_y(t)}{dt}] dt$$

53.3 If we integrate a differential form along two different curves connecting the same endpoints, the value can be different in general.

53.4 If there is a function U such that $A_x = \frac{\partial U}{\partial x}$ and $A_y = \frac{\partial U}{\partial y}$, the differential form A is said to be *exact*: it can be written as $A = dU$. Then

$$\int_\gamma A = U(b) - U(a).$$

The integral of an exact differential form depends only on the endpoints of the curve.

53.5 The *derivative* of a differential 1-form is

$$dA = \frac{1}{2}\left[\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right] dxdy$$

53.6 *Stokes Theorem*: The integral of the derivative of a 1-form on a region D is equal to its integral on the boundary curve γ of the region:

$$\int_D dA = \int_\gamma A$$

53.7 This can be proved for a rectangular region by explicit calculation; more general regions can be broken into rectangular regions and the boundaries can be seen to combine in the required form.

53.8 Suppose that γ_1 and γ_2 are to curves with the same endpoints lying within a disc where $dA = 0$ .Then

$$\int_{\gamma_1} A - \int_{\gamma_2} A = \int_\gamma A = 0$$

Here γ is the closed curve obtained by going along γ_1 first then along γ_2 in the reverse direction.

53.9 In particular, $d[f(z)dz] = \bar\partial f d\bar z dz$; the exterior derivative of the 1-form $f(z)dz$ is zero whenever f satisfies Cauchy-Riemann equations. For example whenever $f(z)$ is analytic.

54 The integral of a function along a closed curve, $\int_\gamma f dz$ vanishes if $\bar\partial f = 0$ within the region bounded by the curve.

54.1 More generally, $\int_D \bar{\partial} f dxdy = \int_\gamma f dz$ if γ is the bounding curve of the region D .

54.2 For example, when $n = 0, 1, 2, 3, \cdots$, $\int_\gamma z^n dz = \int_a^b \gamma(t)^n \frac{d\gamma(t)}{dt} dt = \frac{1}{n+1}[\gamma^{n+1}(b) - \gamma^{n+1}(a)]$ vanishes for closed curves.

54.3 The Cauchy integral theorem allows us to deform contours of integration without changing the integral, provided that the endpoints are unchanged and $\bar{\partial} f = 0$ in the region between the two curves.

54.4 Let S^1 be the *unit circle*; i.e, a circle of unit radius with the origin as the center. Then

$$\int_{S^1} \frac{1}{z} dz = \int_0^{2\pi} e^{-i\theta} i e^{i\theta} d\theta = 2\pi i.$$

This true in fact for a circle of any radius surrounding the origin.

54.5 More generally, along any curve that can be continuosly deformed to unit the circle without crossing the origin the answer is the same.

54.6 The integral $\int_\gamma \frac{1}{z-a} dz$ vanishes if the curve does not enclose the point a : $\bar{\partial} \frac{1}{z-a} = 0$ in the interior.

54.7 Now consider a curve that goes around the point several times: $\gamma(\theta) = e^{in\theta}, n \in Z$. Then, $\int_\gamma \frac{1}{z} dz = 2\pi i n$. In general $\int_\gamma \frac{1}{z-a} \frac{dz}{2\pi i}$ is an integer, the *winding number* of the curve around the point a .

54.8 If a closed curve (assumed to be anti-clockwise) γ bounds a region D , it has winding number one with respect to point in D and zero with respect to any point outside.

54.9 Suppose f is analytic in the region D enlosed by a closed curve γ . If $a \in D$, the function $\frac{f(z)-f(a)}{z-a}$ is also analytic: there is no singularity at a because the numerator also vanishes. Then

$$\int_\gamma \frac{f(z) - f(a)}{z - a} dz = 0.$$

We can re write this result to get a new result.

55 *Cauchy's Integral formula*:If a function f satisfies $\bar{\partial} f = 0$ in the region D bounded by a curve γ ,

$$\int_\gamma \frac{f(z)}{z-a}\frac{dz}{2\pi i} = f(a) \text{ if } a \in D.$$

55.1 This formula gives the solution to the partial differential equation $\bar{\partial} f = 0$ in the region D , in terms of the boundary values of the function on γ .

Chapter 6

Residue Calculus

56 A main application of the Cauchy Integral theorem is to evaluate integrals.

56.1 The integral of a rational function f (whose poles are not on the real line) over the real line is possible by completing the contour on the upper half plane. We assume that he function vanishes at infinity faster than $\frac{1}{z}$, so that, the integral converges. We get

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{\mathrm{Im}a>0} \mathrm{Res}_a f = -2\pi i \sum_{\mathrm{Im}a<0} \mathrm{Res}_a f.$$

56.2 For example,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

57 The same idea applies to meromorphic functions, except that the sum over residues might be an infinite sum.

57.1 The integral of a meromorphic function around a closed contour is $2\pi i$ times the sum of the residues inside the contour.

57.2 For example,

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e}.$$

57.3 As another example, for a natural number n,

$$\int_{|z|=1} e^z z^{-n} dz = 2\pi i \mathrm{Res}_{z=0} e^z z^{-n} = \frac{2\pi i}{[n-1]!}$$

58 Integrals of a periodic function over its period can often be converted into integrals over a circle of a meromorphic function, and then evaluated as above.

58.1 With $z = e^{i\theta}$,

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i \int_{|z|=1} \frac{1}{4z+z^2+1} dz = \frac{2\pi}{\sqrt{3}}.$$

58.2 If the integrand has a simple pole on the contour of integration, the usual definition of the integral will not suffice. Let f be a function which has a simple pole at c, and let $a < c < b$. Then *Cauchy Principal Value integral* is defined as the symmetric limit

$$\mathcal{P}\int_a^b f(x)dx = \lim_{\delta\to 0^+}\left[\int_a^{c-\delta} f(x)dx + \int_{c+\delta}^b f(x)dx\right].$$

58.3 The idea is that the negative infinity from integrating the left side of c is cancelled by the positive infinity from the other side. This idea will not work for higher order poles on the contour of integration.

59 Sums can often be evaluated using residue calculus.

59.1 Let $f(z)$ be a rational function that decays at infinity faster than $\frac{1}{z}$. (That is, the polynomial in the denominator is of degree at least two more than that in the numerator.) Let a_k be the positions of the poles of f and r_k are the residues at these poles. If the poles are not integers,

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum r_k \pi \cot \pi a_k$$

59.2 To prove this, we consider a rectangle C_n with vertices at $\pm[n+\frac{1}{2}]\pm i[n+\frac{1}{2}]$. The function $\pi\cot\pi z$ is bounded on this contour. For big enough z , the function $f(z)$ will be less than a constant times $|z|^{-2}$. Thus

$$\lim_{n\to\infty}\int_{C_n} f(z)\pi\cot\pi z dz = 0.$$

Evaluating the integral by residue calculus, we get the required result. Recall that $\pi\cot\pi z$ has simple poles at the integers, each with residue one.

59.3 For example, if $a>0, b>0$,

$$\sum_{-\infty}^{\infty}\frac{1}{[a+bn^2]} = \frac{\pi}{2\sqrt{[ab]}}\coth[\pi\sqrt{(a/b)}]$$

Chapter 7

The Gamma Function

59.4 Recall that the *factorial* of a natural number is defined as

$$n! = n(n-1)\cdots 1.$$

60 The factorial can be expressed in an integral form

$$n! = \int_0^\infty e^{-t} t^n dt.$$

60.1 To prove this, define

$$I_n = \int_0^\infty e^{-t} t^n dt.$$

It is elementary that $I_0 = 1$. Integration by parts gives the relation

$$I_n = nI_{n-1}.$$

Together they give the required answer. We can even see that the analytical point of view allows us to define

$$0! = I_0 = 1.$$

60.2 The integral converges even for complex values of n as long as the real part is positive. This allows us to define a notion of 'factorial' even for complex numbers.

61 We follow the notation of Euler and define

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \text{ for Re } z > 0.$$

61.1 The integral converges absolutely in the right half plane. Also it converges uniformly on bounded subsets. This means that the integral defines an analytic function. (We just quote this general result about integrals without proof.)

62 For a natural number

$$\Gamma(n+1) = n!.$$

62.1 Again, for $\text{Re} z > 0$, integration by parts gives

$$\Gamma(z+1) = z\Gamma(z).$$

This formula allows us to extend the definition to the left half-plane where the original integral does not converge.

62.2 Thus for example,

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1) = \frac{1}{z} + \Gamma'(1) + O(z).$$

Thus the Γ-function has a simple pole at the origin with residue one. By repeating this argument we get for $n = 0, 1, 2, \cdots$

$$\Gamma(z-n) = \frac{1}{(z-n)(z-n+1)\cdots z}\Gamma(z+1) = \frac{(-1)^n}{n!}\frac{1}{z} + \frac{(-1)^n\Gamma'(1)}{n!} + O(z).$$

Thus at each negative integer $-n$ there is a simple pole of residue $\frac{(-1)^n}{n!}$.

62.3 At other values of the argument the function is regular; using

$$\Gamma(z-n)=\frac{1}{(z-n)(z-n+1)\cdots z}\Gamma(z+1)$$

we can bring the argument to the right half plane. Unless z is a negative integer or zero, the denominator doesnot vanish. Thus $\Gamma(z)$ is a meromorphic function.

63 We have the product relation

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}.$$

63.1 This can be viewed as the limit as $n\to\infty$ of the above recursion relations; however a proof along those lines requires us to know the asymptotic behaviour of $\Gamma(z)$ for large z. A more elementary but indirect argument is to note first that

$$\int_0^n\left[1-\frac{t}{n}\right]^n t^{z-1}dt=\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}$$

and then take the limit as $n\to\infty$.

63.2 In more detail, let $f(n,z)=n^{-z}\int_0^n\left[1-\frac{t}{n}\right]^n t^{z-1}dt$. Integration by parts followed by the change of variable $t=\frac{n}{n-1}u$ gives

$$\begin{aligned} f(n,z) &= n^{-z}\int_0^n\left[1-\frac{t}{n}\right]^{n-1}\frac{t^{(z+1)-1}}{z}dt\\ &= \frac{1}{z}n^{-z}\left[\frac{n}{n-1}\right]^{z+1}\int_0^{n-1}\left[1-\frac{u}{n-1}\right]^{n-1}u^{(z+1)-1}du=\frac{n}{z}f(n-1,z+1). \end{aligned}$$

It follows from the elementary integral $f(1,z)=\frac{1}{z(z+1)}$ and this recursion relation that $n^{-z}\int_0^n\left[1-\frac{t}{n}\right]^n t^{z-1}dt=\frac{n!}{z(z+1)(z+2)\cdots(z+n)}$. Now, we take the limit $n\to\infty$ and use $e^{-t}=\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n$.

64 The *Euler constant* γ is defined as

$$\gamma=\lim_{n\to\infty}\left[\sum_{k=1}^n\frac{1}{k}-\log n\right]\sim 0.577.$$

64.1 The harmonic series $\sum_{k=1}^{n}\frac{1}{k}$ is divergent as can be seeen comparing with the integral $\int_1^n \frac{dx}{x} = \log n$. The difference between the sum and the integral can be written as

$$\sum_{k=1}^{n}\frac{1}{k} - \int_1^n \frac{dx}{x} = \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n-1}\int_k^{k+1}\frac{dx}{x}$$
$$= \frac{1}{n} + \sum_{k=1}^{n-1}\left[\log\left(1+\frac{1}{k}\right) - \frac{1}{k}\right].$$

Now, $\log\left(1+\frac{1}{k}\right) - \frac{1}{k}$ is bounded by a constant times $\frac{1}{k^2}$; hence the sum on the rhs has a limit as $n \to \infty$. Thus

$$\gamma = \sum_{k=1}^{\infty}\left[\log\left(1+\frac{1}{k}\right) - \frac{1}{k}\right]$$

exists.

65 Thus, $\frac{1}{\Gamma(z)}$ is an entire function with the product representation

$$\frac{1}{\Gamma(z)} = z e^{\gamma z}\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\right].$$

65.1 The product $\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\right]$ converges but not $\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)$, since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges. When the extra factor $e^{-\frac{z}{k}}$ is put in the large k behavior is improved: $\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} \sim 1 - \left[\frac{z}{k}\right]^2$.

65.2 To prove the result above, we use the product formula:

$$\frac{1}{\Gamma(z)} = z\lim_{n\to\infty}\frac{(z+1)(z+2)\cdots(z+n)}{n!n^z}$$
$$= z\lim_{n\to\infty}(1+z)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right)e^{-z\log n}$$
$$= z\lim_{n\to\infty}(1+z)e^{-z}\left(1+\frac{z}{2}\right)e^{-\frac{z}{2}}\cdots\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\,e^{z\left[\left(1+\frac{1}{2}+\cdots\frac{1}{n}\right)-\log n\right]}$$

66 Also, we have the ***reflection formula***

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}.$$

66.1 For this we need the product formula for the trigonometric function

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left[1 - \left(\frac{z^2}{n^2}\right)\right].$$

The various exponentials cancel out when we mutiply the product representations of $\frac{1}{\Gamma(z)}$ and $\frac{1}{\Gamma(-z)}$. To complete the proof of the reflection formula we also use $\Gamma(1-z) = -z\Gamma(-z)$.

66.2 The logarithmic derivative of a meromorphic function is also meromorphic. In our case this is the function (again in the notation of Euler)

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

This is also called the *digamma* function. The logarithmic derivative of the recursion formula for $\Gamma(z)$ gives

$$\psi(z+1) = \frac{1}{z} + \psi(z).$$

66.3 Recall that wherever $f(z)$ has a zero (or pole), $\frac{f'(z)}{f(z)}$ will have a simple pole of residue equal to the (negative) of the order of the zero (or pole). Thus $\psi(z)$ has poles at negative integers or zero with residue -1 . This is evident by repeatedly using the recursion formula above.

67 Many physical problems require us to know the behaviour of the Gamma function for large values of its argument.

Chapter 8

Asymptotic expansion of Integrals

68 We will now study the behavior of

$$Z(\lambda) = \int_a^b e^{-\frac{1}{\lambda}S(x)} dx,$$

for small λ .

68.1 Here, $S(x)$ is given as a function of x ; in many interesting cases it is a polynomial.

68.2 The classics *Priciples of Optics* by M. Born and E. Wolf and *Optical Coherence and Quantum Optics* by L. Mandel and E. Wolf are good references-but they consider the subtler case of imaginary λ . See also Chapter six of the book by Bender and Orszag for another point of view. The discussion below does not follow either book.

68.3 Such problems arise in every branch of physics: the geometrical limit of wave optics, the classical limit of quantum mechanics, the mean field limit of statistical mechanics, the Feynman-Dyson expansion of quantum field theory, the Stirling formula for the Gamma function, are all examples of this problem.

68.4 We will study here only the case of an integral in one variable; moreover we will assume for simplicity that S is real-valued , and has a unique minimum within the interval $[a, b]$ These are not essential assumptions; it is possible to generalize the method much further.

68.5 The basic idea is that when λ is real positive and small, only the smallest values of S matter. Then we expand the it around the minimum of S in a power series. We isolate a piece that has an essential singularity at $\lambda = 0$. (This is the tricky part.) Then we expand the rest as a power series in λ .

68.6 Let $a < x_0 < b$ be the minimum:

$$S'(x_0) = 0, \quad S''(x_0) > 0, \Rightarrow S(x) = S(x_0) + \frac{1}{2}(x - x_0)^2 S''(x_0) + \frac{1}{3!} S'''(x_0)(x - x_0)^3 + \cdots$$

Then,

$$Z(\lambda) = e^{-\frac{1}{\lambda}S(x_0)} \int_a^b dy e^{-[\frac{1}{2\lambda}S''(x_0)[x-x_0]^2 + \frac{1}{3!\lambda}S'''(x_0)[x-x_0]^3 + \cdots]}$$

68.7 Making the change of variables

$$y = \left[\frac{S''(x_0)}{\lambda}\right]^{\frac{1}{2}} (x - x_0)$$

we get (with $g = \left[\frac{\lambda}{S''(x_0)}\right]^{\frac{1}{2}}$),

$$Z(\lambda) = e^{-\frac{1}{\lambda}S(x_0)} \left[\frac{2\pi\lambda}{S''(x_0)}\right]^{\frac{1}{2}} \int_{\frac{a-x_0}{g}}^{\frac{b-x_0}{g}} dy e^{-[\frac{1}{2}y^2 + \frac{1}{3!}gS'''(x_0)y^3 + \frac{1}{4!}g^2 S''''(x_0)y^4 + \mathrm{O}(g^3)]}$$

68.8 As $\lambda \to 0^+$, the limits tend to $\pm\infty$; the error in replacing the integrals with integrals over $\pm\infty$ is of relative order $e^{\frac{-1}{\lambda}}$

$$\begin{aligned} Z(\lambda) &\sim e^{-\frac{1}{\lambda}S(x_0)} \left[\frac{\lambda}{S''(x_0)}\right]^{\frac{1}{2}} \int_{-\infty}^{\infty} dy e^{-[\frac{1}{2}y^2 + \frac{1}{3!}gS'''(x_0)y^3 + \frac{1}{4!}g^2 S''''(x_0)y^4 + \mathrm{O}(g^3)]} \\ &\sim e^{-\frac{1}{\lambda}S(x_0)} \left[\frac{\lambda}{S''(x_0)}\right]^{\frac{1}{2}} \int_{-\infty}^{\infty} dy e^{-\frac{1}{2}y^2} [1 - \frac{1}{3!}gS'''(x_0)y^3 - \frac{1}{4!}g^2 S''''(x_0)y^4 + \\ &\quad \frac{1}{2(3!)^2} g^2 [S''']^2 y^6 + \mathrm{O}(g^4)] \\ &\sim e^{-\frac{1}{\lambda}S(x_0)} \left[\frac{2\pi\lambda}{S''(x_0)}\right]^{\frac{1}{2}} \left[1 - \frac{1}{8}g^2 S''''(x_0) + \frac{5}{24}g^2[S'''(x_0)]^2 + \mathrm{O}(g^4)\right] \end{aligned}$$

68.9 Here we use

$$\int e^{-\frac{1}{2}x^2} x^{2n} dx = \surd[2\pi][2n-1]!!.$$

This lemma can be proved by an integration by parts leasing to an induction on n . The case $n = 0$ can be determined by relating the integral to $\Gamma(\frac{1}{2})$.

69 An application is to determine the behavior of $\Gamma(z)$ for large $|z|$.

69.1 Substituting $t = xz$ in the definiing integral of the Gamma function,

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = \int_0^\infty e^{-[t - z\log t]} dt = z e^{z \log z} \int_0^\infty e^{-z[x - \log x]} dx$$

69.2 The integral is of the form we studied above with $z = \frac{1}{\lambda}$, $S(x) = x - \log x$. The minimum (in the interval $[0, \infty]$) of $S(x)$ occurs at $x_0 = 1$ so that $S(x_0) = 1, S''(x_0) = 1, S'''(x_0) = 2, S''''(x_0) = 3!$ etc. Thus

$$\int_0^\infty e^{-z[x-\log x]} dx = e^{-z} \left[\frac{2\pi}{z}\right]^{\frac{1}{2}} \left[1 + \frac{1}{12z} + \cdots\right]$$

69.3 Putting it all together

$$\Gamma(z+1) = [2\pi]^{\frac{1}{2}} z^{z+\frac{1}{2}} e^{-z} \left[1 + \frac{1}{12z} + \cdots\right]$$

69.4 For example, $10! = 3628800$. The first term in the above approximation gives 3598695.6 ; including the second term gives the improved estimate 3628684.7 .

69.5 The above expansion for the factorial is supposed to work for large values of z . But it is unreasonably effective even for small values of z . For example, the exact value of $2! = 2$ while the first term in the above expansion gives 1.919 ; including the next term gives 1.99896 . This has important implications in physics. Many approximation schemes (such as the semi-classical expansion in quantum mechanics, the eikonal expansion in wave optics, the $\frac{1}{N}$ expansion in Yang-Mills theories) are more sophisticated versions of the basic method described in this chapter. These approximations are 'unreasonably effective' as well.

69.6 Carrying this procedure out to higher orders we get the *Stirling expansion*

$$\Gamma[z+1] = [2\pi]^{\frac{1}{2}} z^{z+\frac{1}{2}} e^{-z} \left[1 + \frac{1}{12\,z} + \frac{1}{288\,z^2} - \frac{139}{51840\,z^3} - \frac{571}{2488320\,z^4} + \frac{163879}{209018880\,z^5} + \mathrm{O}(\frac{1}{z^6})\right]$$

69.7 There are special tricks that allows us to derive the Stirling expansion for the Gamma function (see *Ahlfors*) itself in a simpler way. The method we give is generic- it is at the heart of most approximation methods in physics.

Chapter 9

Differential Equations in Physics

70 Except in Quantum Field Theory, all the laws of physics are expressed in terms of differential equations. Solving differential equations is central to physics.

70.1 Quantum Field Theory is the theory of elementary particles and of systems with an infinite number of particles. It can be formulated as differential equations in an infinite number of variables-the Schwinger-Dyson equations.

71 The laws of classical mechanics are second order ordinary differential equations .

71.1 Newton's Laws state that

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$$

where $\mathbf{x}(t)$ is the position of a particle as a function of time, m its mass and $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$ the force as a function of position, velocity and time. This is usually a system of nonlinear ordinary differential equations. In very simple cases, the force may depend linearly on the position and velocity and the equations become linear.

71.2 Another formulation is in terms of Hamilton's equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

Here q_i for $i = 1, \cdots n$ are the position variables and p_i the momentum variables; also, $H(q, p, t)$ is the *hamiltonian*.

71.3 These are also a system of nonlinear ordinary differential equations.

72 Classical mechanics can also be expressed in terms of first order partial differential equations.

72.1 These are the *Hamilton-Jacobi equations*

$$H\left(q, \frac{\partial W}{\partial q}, t\right) = \frac{\partial W}{\partial t},$$

where $W(q, t)$ is the *action*.

73 The fundamental laws of classical electromagnetism are partial differential equations, the *Maxwell equations*.

73.1

$$\nabla \cdot E = 4\pi\rho, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

Here, $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{j}$ the electric current, and ρ the charge density.

73.2 Here, c is the velocity of light.

74 Electrostatics is described by the Laplace and Poisson equations.

74.1 If the electric field is independent of time and the charges are not moving (there is no current), the electric field is the gradient of the **electrostatic potential**, $\mathbf{E} = -\nabla V$ .Then

$$\nabla^2 V = \rho$$

which is *Poisson's equation*. If there are no electric charges, we get the *Laplace equation*:

$$\nabla^2 V = 0.$$

74.2 Solving Laplace's equation in various geometries is one of the joys(??) of a physics education.

74.3 The Lapace equation is an example of an elliptic PDE; they describe static (time independent) phenomena.

75 The wave equation of optics follows from Maxwell's equations.

75.1 Every phenomenon of classical optics follows from the wave equation of the electromagnetic field, which in turn is a consequence of the Maxwell equations.

75.2 The simplest version of the wave equation is

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0.$$

This describes waves of a single polarization propagating in the vacuum.

75.3 We have chosen natural units in which the speed of light is one.

75.4 The wave equation is the prototype of a hyperbolic PDE; they describe time dependent phenomena, where signals are propagated by some wave.

76 *Geometrical optics* is the limit of small wavelength. It is described by either the *eikonal equation* (a first order PDE) or the characteristic equations for light rays (a nonlinear ODE).

77 The basic law of quantum mechanics is a second order linear partial differential equation: the Schrodinger equation. All of chemistry (and perhaps biology) follows from quantum mechanics.

77.1 For a system of particles with hamiltonian $H(q, p, t)$, the Schrodinger equation is

$$H\left(q, \frac{\hbar}{i}\frac{\partial}{\partial q}, t\right)\psi = i\hbar\frac{\partial \psi}{\partial t}.$$

$\psi(q, t)$ is the *wavefunction*, a complex-valued function of position and time.

77.2 Here, $\hbar$ is *Plank's constant*; sometimes $h = 2\pi\hbar$ is called Planck's constant instead.

77.3 Only for the very simplest systems can this equation be solved. Most of physics is about approximation methods. In the classical limit of small $\hbar$, $\psi \sim e^{\frac{i}{\hbar}W}$, the Schrodinger equation reduces to the Hamilton-Jacobi equation.

78 The equations of a fluid are

$$\frac{\partial[\rho\mathbf{u}]}{\partial t} + \mathbf{u}\cdot\nabla[\rho\mathbf{u}] = -\nabla p + \eta\nabla^2\mathbf{u} + \mathbf{f}, \quad \nabla\cdot[\rho\mathbf{u}] + \frac{\partial\rho}{\partial t} = 0.$$

78.1 Here ρ is density; the p the pressure is assumed to be known as a function of the density through an equation of state. Also, η is the viscocity of the fluid.

78.2 If the velocity is small compared to the speed of sound (*subsonic*), the flow is *incompressible*: ρ becomes a constant.

79 Subsonic fluid flow is determined by the *Navier-Stokes* equations, a system of parabolic PDE.

79.1 They follow from Newton's laws applied to a system of with a large number of particles-the molecules of the fluid.

79.2

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0.$$

$\mathbf{u}(\mathbf{x}, t)$ is the velocity as a function of position and time; p is the *pressure*, ν a property of the fluid called *kinematic viscosity*, and $\mathbf{f}$ the external force that might act on a fluid element.

79.3 Understanding the phenomenon of *turbulence*, which follows from these equations is the third deepest problem in all of physics.

79.4 Parabolic equations describe dissipative systems. The dissipation of velocity gradients due to viscosity prevents the fluid flow from becoming too turbulent.

79.5 A simpler example of a parabolic PDE is the diffusion equation

$$\frac{\partial P}{\partial t} = \nabla^2 P.$$

It describes such dissipative phenomena as diffusion, heat conduction or random walks.

79.6 The limit of zero viscosity (*ideal fluid*) and no external forces is of special interest, because then, turbulence is dominant. In this limit the Navier-Stokes equations reduce to the *Euler equations*:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p, \quad \nabla \cdot \mathbf{u} = 0.$$

79.7 The Euler equations describe geodesics on the group of volume preserving diffeomorphisms.

80 The three deepest theories of physics are Einstein's theory of gravity (general relativity), the Yang-Mills theory of strong interactions (quantum chromodynamics) and this Euler theory of the ideal fluid (turbulence).

81 Plasma physics is described by the Vlasov equations.

81.1 Plasma physics is a combination of fluid mechanics and electromagnetism-the source for the electromagnetic field is a fluid.

82 Discovering the fundamental equations of nature is merely the beginning of the study of physics.

83 Theoretical physics is the solution of the fundamental equations of nature in various approximations.

83.1 Only the simplest, most idealized systems, are exactly solvable by analytic methods.

83.2 The fundamental physical laws are the alphabet in the language of nature. These laws in various combinations form the literature of natural phenomena.

83.3 Analytic methods aim to give a qualitative and fundamental understanding of physical phenomena.

83.4 Detailed quantitative study of systems is usually based on numerical simulations on digital computers.

Chapter 10

Linear ODE with Constant Coefficients

84 An *ordinary differential equation* or *ODE* is a condition on a function $y : C \to C$ and its derivatives $F(x, y, y', y'', \cdots y^{(n)}) = 0$.

84.1 If F depends on the first n derivatives of y only, we say that the equation is of *order* n .

84.2 All of the following equations arise from various problems in physics:

$$\begin{aligned} y' - y &= 0 \\ y'' + y &= 0 \\ \frac{1}{2}y'^2 + \frac{1}{2}x^2 y &= c \\ y'' + by' + ky &= 0 \\ y'' + xy &= 0. \end{aligned}$$

84.3 Thus the unknown quantity is a function rather than a number; thus the theory of ODE is a step higher than that of algebraic equations. There is no general method to solve ODE exactly: only the simplest equations can be solved analytically.

84.4 There are general methods that solve an ODE with boundary conditions numerically. With the use of digital computers, a whole world has opened up to these methods in the last fiftly years.

85 An ODE has usually many solutions.

85.1 To determine the solution we need additional information in the form of *boundary conditions*: the value of y or its derivatives at some values of x .

86 If the condition depends linearly on y and all its derivatives, the equation is said to be *linear*.

87 If $y = 0$ is a solution, an ODE is said to be *homogenous*.

87.1 The set of solutions of a homogenous linear ODE form a vector space: the sum of two solutions as well as a constant times a solutions are still solutions.

88 Homogenous linear ODE with constant coefficients can often be solved by a finite series of exponential functions.

88.1 For example,

$$y'' + 3y = 0$$

has a solution $y(x) = e^{\omega x}$ provided that $\omega^2 + 3 = 0$; thus the general solution is

$$y(x) = C_1 e^{i\sqrt{3x}} + C_2 e^{-i\sqrt{3x}}.$$

The constants C_1, C_2 are determined from the boundary conditions. For example, if we are given that $y(0) = 1, y'(0) = 0$ we have $C_1 + C_2 = 1, C_1 - C_2 = 0$ so that $y(x) = \cos[\sqrt{3}x]$.

88.2 Let us consider the general homogenous linear ODE with constant coefficients:

$$\sum_{r=0}^{n} a_r \frac{d^r y}{dx^r} = 0;$$

if $a_n \neq 0$ this is of order n . The *ansatz* $y(x) = e^{\omega x}$ gives

$$\sum_{r}^{n} a_r \omega^r = 0.$$

Thus we have reduced the ODE to an algebraic equation of order n . By the fundamental theorem of algebra, there are n solutions, some of which may coincide.

88.3 If there are n distinct solutions, the general solution of the ODE is

$$y(x) = \sum_{r=1}^{n} C_r e^{\omega_r x}.$$

88.4 To understand the case when the roots coincide, let us look at the equation

$$y'' + 2y' + y = 0$$

which yields the associated algebraic equation

$$\omega^2 + 2\omega + 1 = 0$$

with the root $\omega = -1$ with multiplicity two. Thus $y(x) = e^{-x}$ is a solution. There is one more solution for the ODE, which our exponential ansatz misses: $y(x) = xe^{-x}$.

88.5 Think of the case with a double root as a limit when two roots coincide. The general solution when $\omega_1 \neq \omega_2$ is

$$y(x) = C_1 e^{\omega_1 x} + C_2 e^{\omega_2 x}.$$

The constants C_1, C_2 are determined by the boundary conditions. For example, if $y(0), y'(0)$ are given,

$$y(0) = C_1 + C_2, \quad y'(0) = \omega_1 C_1 + \omega_2 C_2.$$

Thus, $y'(0) = \omega_1 y(0) + [\omega_2 - \omega_1]C_2$. Thus, in the limit $\omega_2 \to \omega_1$, we must hold $C_2' = [\omega_2 - \omega_1]C_2$, rather than C_2 itself, fixed. Now, expanding in powers of $\omega_2 - \omega_1$, we get

$$y(x) = y(0)e^{\omega_1 x} + C_2' x e^{\omega_1 x}$$

since the remaining terms vanish in the limit. Thus a basis for the solutions in the case of coincident roots is given by $e^{\omega_1 x}, xe^{\omega_1 x}$.

88.6 The same idea shows that if we have a root of multiplicity m , $e^{\omega}, xe\omega x, \cdots x^{m-1}e^{\omega x}$ are linearly independent solutions.

89 A homogenous linear ODE of order n , with constant coefficients:

$$\sum_{j=0}^{n} a_j \frac{d^j y}{dx^j} = 0$$

has n independent solutions. If the associated algebraic equation

$$\sum_{j=0}^{n} a_j \omega^j = 0$$

has solutions

$$\omega_1, \cdots \omega_k$$

of multiplicities m_1, m_2, m_k respectively, then the general solution of the ODE is

$$y(x) = \sum_{r=1}^{k} \sum_{s=0}^{m_r - 1} C_{rs} x^s e^{\omega_r x}.$$

89.1 Since $\sum_{r=1}^{k} m_r = n$ by the fundamental theorem of algebra, there are n constants that have to be fixed by boundary conditions.

89.2 In other words, the set of solutions of a homogenous linear equations is a vector space of dimension n ; a basis for this vector space is given by $x^s e^{\omega_r x},$.

90 The translational symmetry of homogenous linear ODE suggests the exponential ansatz.

90.1 A homogenous linear ODE with constant coefficients is invariant under the transformation $x \to x + a$ for any constant a ; hence, if $y(x)$ is a solution, so is $y(x+a)$. The simplest way this can happen is that $y(x+a)$ is a constant times $y(x)$. This suggests the exponential ansatz: $e^{\omega[x+a]} = e^{\omega a} e^{\omega x}$.

90.2 In general, $y(x+a)$ will be a linear combination of solutions. If $y_j(x)$ is a basis of solutions, then each $y_j(x+a)$ must be a linear combination: $y_j(x+a) = \sum_k T(a)_{jk} y_k(x)$. A matrix can be brought to a Jordan canonical form by a choice of basis. If it can be made diagonal, the roots are distinct; in general it can be brought to a block diagonal form, with triangular matrices for each root.

91 If a homogenous linear equation is invariant under the scale transformation $x \to \lambda x$ the ansatz $y(x) \sim x^\alpha$ will reduce it to an algebraic equation.

91.1 For example,

$$x^2 y'' - 2xy' + 2y = 0$$

yields, with $y(x) = x^\alpha$,

$$\alpha(\alpha - 1) - 2\alpha + 2 = 0 \Rightarrow \alpha = -1, -2$$

so that $y(x) = C_1 x^{-1} + C_2 x^{-2}$ is the general solution.

91.2 The point is that in terms of the variable $t = \log x$ such equations have constant coefficients.

91.3 If the associated algebraic equation has a double root α , then x^α and $x^\alpha \log x$ both solve the ODE.

92 The ODE

$$\sum_{j=0}^{n} a_j x^j \frac{d^j y}{dx^j} = 0$$

has the associated algebraic equation

$$\sum_j^n a_j \alpha(\alpha - 1) \cdots (\alpha - j + 1) = 0.$$

If its roots are $\alpha_1 \cdots \alpha_k$ with multiplicities $m_1, \cdots m_k$, the general solution of the ODE is

$$y(x) = \sum_{r=1}^{k} \sum_{s=0}^{m_r - 1} C_{rs} \left[\log x\right]^s x^{\alpha_r}.$$

93 A set of functions $\{f_1, f_2 \cdots f_n\}$ **is** ***linearly independent*** **if** $\sum_{j=1}^{n} C_j f_j(x) = 0$ **for all** x **implies** $C_1 = C_2 = \cdots C_n = 0$ **.**

94 A necessary and sufficient condition for linear dependence is that the ***Wronskian*** **of the functions,**

$$\det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ . & . & \cdots & . \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}$$

be zero for every x **.**

94.1 Thus if the Wronskian is non-zero at even one point, the functions are linearly independent.

94.2 For example, $\{e^x,\ xe^x\}$ is a linearly independent set.

Chapter 11

The Hypergeometric Function

95 Solvable problems of physics lead to functions that are special cases of the hypergeometric function.

95.1 These are the so called Higher Transcendental Functions- the exponential function,Bessel functions, Error function, Hermite polynomials, Laguerre polynomials, etc. are examples.

95.2 The vasy majority of differential equations of physics are not solvable; yet the solvable ones are precious, as they are the starting point of many approximation methods.

96 Recall the geometric series and the exponential:

$$\begin{aligned}\frac{1}{1-z} &= 1 + z + z^2 + z^3 + \cdots \\ e^z &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\end{aligned}$$

96.1 The exponential is obtained by inserting an extra factor of $\frac{1}{n!}$ into the denominator of each term of the geometric series. This is the starting point of a family of functions called the hypergeometric functions.It will turn out to be convenient to think of it as the zeroth element of this family: $F_0^0(z) = e^z$.

96.2 We have then $\frac{d}{dz}e^z = e^z$.

97 The next member of the hypergeometric family of functions is

$$F_1^0(\gamma; z) = 1 + \frac{1}{\gamma}\frac{z}{1!} + \frac{1}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{1}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \cdots$$

for γ not equal to a negative integer or zero.

97.1 The Bessel functions are special cases of this function.

97.2 We can regard the exponential as a limiting (or *confluent*) case:

$$\lim_{\gamma\to\infty} F_1^0(\gamma; \gamma z) = F_0^0(z).$$

97.3 You can verify by differentiating term by term that

$$\frac{d}{dz}F_1^0(\gamma; z) = \frac{1}{\gamma}F_1^0(\gamma+1; z).$$

97.4 Also,that it is one solution to the ODE

$$zu'' + \gamma u' - u = 0.$$

97.5 Bessel's equation

$$zy'' + zy' + [z^2 - \nu^2]y = 0$$

reduces to

$$zf'' + (2\nu+1)f' + zf = 0$$

with $y(z) = z^\nu f(z)$. This is invariant under $z \to -z$. So there is a solution in even powers of z , suggesting the substitution $t = cz^2$ for some constant c . We will get the above equation for $F_1^0(\gamma; z)$; i.e., $t\ddot{u} + [\nu+1]\dot{u} - u = 0$ with the choice $t = -\frac{1}{4}z^2$.

98 The confluent hypergeometric series with two parameters is

$$F_1^1(\alpha,\gamma;x) = 1 + \frac{\alpha}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots$$

98.1 The one with a single parameter is a limiting case

$$F_1(\beta;t) = \lim_{\alpha\to\infty} F_1^1(\alpha,\beta,\frac{t}{\alpha})$$

because then

$$\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)\left[\frac{t}{\alpha}\right]^k = (1+\frac{1}{\alpha})(1+\frac{2}{\alpha})(1+\frac{3}{\alpha})\cdots(1+\frac{k-1}{\alpha})t^k \to t^k.$$

98.2 The equation satisfied by $F_1^1(\alpha,\gamma,x)$ is

$$xu'' + [\gamma - x]u' - \alpha u = 0.$$

If we set $x = \frac{t}{\alpha}$

$$t\ddot{u} + [\gamma - \frac{t}{\alpha}]\dot{u} - u = 0$$

Now let $\alpha \to \infty$ this will reduce to the equation for $F_1(\gamma,t)$:

$$t\ddot{u} + \gamma\dot{u} - u = 0.$$

98.3 There is a direct connection of Bessel's equation to the confluent hypergeometric series F_1^1 :

$$J_\nu(z) = Cz^\nu e^{-iz}F_1^1(\frac{1}{2}+\nu, 1+2\nu, 2iz).$$

The constant $C = \frac{1}{\Gamma(\nu+1)}2^{-\nu}$ is needed to match with the usual conventions for Bessel's functions in most textbooks.

98.4 If we start with Bessel's equation

$$z^2y'' + zy' + [z^2 - \nu^2]y = 0$$

and make the substitution $y = z^\nu e^{-iz} f(z)$ (suggested by the behavior at infinity and the origin) we get an ODE for $f(z)$ of the confluent hypergeomeric form.

99 In fact F_1^1 is itself a limiting case of the hypergeometric series

$$F_1^2(\alpha, \beta, \gamma; z) = 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \cdots$$

100 F_1^2 satisfies the ODE

$$z(1-z)u'' + [\gamma - (\alpha+\beta+1)z]u' - \alpha\beta u = 0.$$

100.1 This can be verified by term by term differentiation.

100.2 By the same argument as above,

$$F_1^1(\alpha, \gamma; z) = \lim_{\beta\to\infty} F_1^2(\alpha, \beta; \gamma, \frac{z}{\beta}).$$

100.3 Thus F_1^2 is the mother of all such series. We can trace the genealogy even further: generalizing to the the hypergeometric series with parameters $\alpha_1 \cdots \alpha_p$ in the numerator and q such parameters in the denominator. But you have probably had enough! If not you can read a bit further.

101 The generalized hypergeometric series F_q^p is the formal power series

$$F_q^p(\alpha_1 \cdots \alpha_p; \gamma_1 \cdots \gamma_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\gamma)_1(\gamma_2)_n \cdots (\gamma_q)_n}\frac{z^n}{n!}.$$

101.1 Here we use the abbreviation $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+n-1)$.

101.2 We are of course assuming that none of the $\gamma_1 \cdots \gamma_q$ are not equal to negative integers or zero.

101.3 If $p \leq q+1$ this series will have non-zero radius of convergence.In general it may not converge anywhere and should be thought of as a formal pwer series.

101.4 The ratio of successive coefficients of z^n is a rational function of n ; conversely, any series whose ratios of successive coefficients are rational functions of n can be expressed as linear coefficients of the F_q^p .

101.5 More explicity, if we write $F_q^p(\alpha;\gamma;z) = \sum_{n=0}^{\infty} a_n z^n$,

$$\frac{a_{n+1}}{a_n} = \frac{P(n)}{Q(n+1)}$$

we have

$$P(n) = (\alpha_1+n)\cdots(\alpha_p+n)(\gamma_1+n-1)\cdots(\gamma_q+n-1).$$

$$Q(n+1) = (n+1)(\gamma_1+n)\cdots(\gamma_q+n)(\alpha_1+n-1)\cdots(\alpha_p+n-1).$$

102 The generalized hypergeometric series satisfy the ODE

$$[zP(\theta) - Q(\theta)]y = 0$$

where $\theta = z\frac{d}{dz}$ **.**

102.1 The point is that the effect of $z\frac{d}{dz}$ on $\sum_n a_n z^n$ is to replace a_n by na_n . Thus the ODE leads to the recursion relation

$$\frac{a_{n+1}}{a_n} = \frac{P(n)}{Q(n+1)}.$$

102.2 Any ordinary differential operator can be thought of as a polynomial in $\theta = z\frac{d}{dz}$ with coefficients that are functions of z . If these coefficients can be chosen to be linear functions of z , the differential equation can be brought to the hypergeometric form. (Another way to say this is that the recursion relation for the coefficients be first order.) Even more generally, if the coefficients of θ are polynomials in z , (so that the recursion relation may be higher than first order) we can reduce it to a limiting case (letting some of the α, γ diverge) of some hypergeometric equation.

103 All ODE with rational coefficients can be solved in terms of hypergeometric series.

103.1 We must express the differential operator as a polynomial in θ with coefficients that are themselves polynomials in z . This is always possible by multiplying through by some common factor. Then the recursion relation for the coefficients of z^n can be derived; the series so obtained can then be identified as a hypergeometric series (or a limiting case of one).

103.2 For a more complete theory, see *Higher Transcendental Functions* by Bateman edited by Erdelyi.

Chapter 12

A Nonlinear ODE

104 There is no general method to solve nonlinear ODE.

105 An active research area called *chaos* has emerged from the study of nonlinear ODE.

106 Simple ODE can often be solved by making changes of variables.

107 ***An airplane moves at a constant velocity v , at a fixed altitude h . A heat-seeking missile is launched when the airplane is directly overhead. The missile has a constant speed u and always moves in a direction pointed directly at the airplane. Find the curve followed by the missile.***

108 The geometry of the problem can be translated into an ODE.

108.1 We get the differential equations

$$\dot{x}^2 + \dot{y}^2 = v^2, \quad \frac{\dot{x}}{\dot{y}} = \frac{ut - x}{h - y}.$$

Also, we have the initial conditions $x = y = 0 = \dot{x}$ at $t = 0$.

109 *Autonomous equations*–ODE in which the independent variable does not directly appear–are easier to study.

109.1 If we trade x for the the position as measured by an observer in the airplane, $\xi = ut - x$ we can get rid of the explicit time dependence in the equation:

$$[u - \dot{\xi}]^2 + \dot{y}^2 = v^2, \quad u - \dot{\xi} = \eta \dot{y}$$

where

$$\eta = \frac{\xi}{h - y}$$

is the inverse of the slope of the curve.

110 Sometimes we should use some other variable as the independent variable.

110.1 Solving for $\dot{\xi}, \dot{y}$,

$$\dot{y} = \frac{v}{\sqrt{[1 + \eta^2]}}, \quad \dot{\xi} = u - \frac{v\eta}{\sqrt{[1 + \eta^2]}}.$$

Thus $\frac{d\xi}{dy} = \frac{\dot{\xi}}{\dot{y}}$ can be expressed in terms of η alone. So we should use y as the independent variable instead of t .

110.2 Differentiating the relation $\eta(h - y) = \xi$ gives

$$(h - y)\frac{d\eta}{dy} - \eta = \frac{d\xi}{dy} = \frac{u}{v}\sqrt{[1 + \eta^2]} - \eta.$$

In other words

$$(h - y)\frac{d\eta}{dy} = \frac{u}{v}\sqrt{[1 + \eta^2]}.$$

110.3 This is a separable equation:

$$\int_0^\eta \frac{d\eta}{\sqrt{[1 + \eta^2]}} = \frac{u}{v}\int_0^y \frac{dy}{h - y}.$$

Or,

$$\eta = \frac{1}{2}\left[\left(\frac{h}{h - y}\right)^\alpha - \left(\frac{h}{h - y}\right)^{-\alpha}\right], \quad \alpha = \frac{u}{v}.$$

At this point we have solved the ODE; what remains is to backtrack along the changes of variables we made to determine the path of the missile.

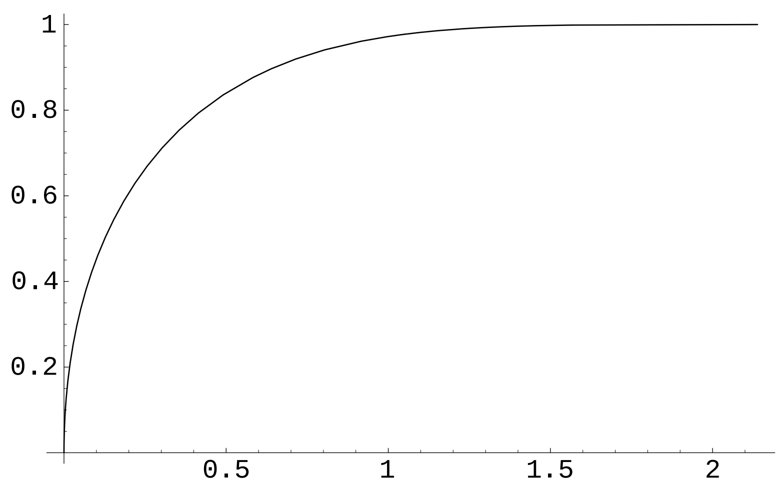
110.4 Since $\eta = \frac{dx}{dy}$ this can be integrated to determine x as a function of y - now we have the path of the missile:

$$x(y) = \frac{-2\,\alpha\,h + h\,\left(-1 + \alpha + \left(\frac{h}{h-y}\right)^{2\,\alpha} + \alpha\left(\frac{h}{h-y}\right)^{2\,\alpha}\right)\,\left(\frac{h}{h-y}\right)^{-1-a}}{2\,\left(-1+\alpha^2\right)}$$

110.5 For the missile to hit the airplane, we need $v > u$ or equivalently $\alpha < 1$. The distance the plane travels before it is hit by the missile is the limiting value as $y \to h$ of the above:

$$X = \frac{h\alpha}{1-\alpha^2}.$$

110.6 Here is the path of the missile when $\alpha = 0.8$; we measure the distances in units of h .



110.7 You can complete the story by even finding the time dependence of y .We now know $\dot{y}$ as a function of y . So,

$$t = \int_0^y \frac{1}{\dot{y}} dy = \frac{1}{v}\int_0^y \sqrt{[1+\eta(y)^2]}dy$$

which can be done by mathematica:

$$t = \frac{h[f_1(y)+f_2(y)+f_3(y)]}{2\,(-1+\alpha)\,(1+\alpha)\,\left(1+\left(\frac{h}{h-y}\right)^{2\,\alpha}\right)}$$

where

$$f_1(y) = \left(-2+\sqrt{2+\left(\frac{h}{h-y}\right)^{-2\,\alpha}+\left(\frac{h}{h-y}\right)^{2\,\alpha}}\right)\left(1+\left(\frac{h}{h-y}\right)^{2\,\alpha}\right)$$

$$f_2(y) = \alpha\,\left(-1+\left(\frac{h}{h-y}\right)^{2\,\alpha}\right)\sqrt{2+\left(\frac{h}{h-y}\right)^{-2\,\alpha}+\left(\frac{h}{h-y}\right)^{2\,\alpha}}$$

$$f_3(y) = -\left(1+\alpha\,\left(-1+\left(\frac{h}{h-y}\right)^{2\,\alpha}\right)+\left(\frac{h}{h-y}\right)^{2\,\alpha}\right)\sqrt{2+\left(\frac{h}{h-y}\right)^{-2\,\alpha}+\left(\frac{h}{h-y}\right)^{2\,\alpha}}$$

This determines the time dependence of the path parametrically.

111 The key steps where to remove the time dependence by moving to the reference frame of the airplane; and to regard y as the independent variable instead of t .

Chapter 13

The Wave Equation

112 All stable mechanical systems oscillate around their equilibrium configuration.

112.1 At an equilibrium, the force acting on the system is zero. A small displacement from equilibrium might cause a force that pushes away from the equilibrium point- then we have an unstable equilibrium. If every small displacement leads to a force that acts to reduce the displacement, we have stable equilibrium point. In this case the system will go back to the equilibrium point, possibly overshoot it then be pushed back again and so on. In the absence of dissipation (friction etc.) these oscillations can go on for ever. But typically they die down eventually.

112.2 A mechanical system whose configuration at an instant is determined by a function of space is called a *field*. Examples are the height of the surface of a liquid, the pressure of a gas and the eletromagnetic field.

113 The oscillations of a field are waves.

113.1 A small disturbance on the surface of water causes waves (ripples) to form; they spread until they are reflected by the boundary. The oscillations in the pressure of a gas is *sound*. Oscillations of the electromagnetic field of certain frequencies is percieved as *light*.

114 The wave equation in one spatial dimension is

$$\frac{\partial^2}{\partial t^2}\phi - \frac{\partial^2}{\partial x^2}\phi = 0.$$

114.1 If we define the variables $u = x + t, v = x - t$, this can be brought to the form

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0.$$

Thus any function of u alone or v alone will satisfy this equation.

115 The general solution of this wave equation is

$$\phi(t, x) = f(x + t) + g(x - t).$$

115.1 If we are given initial conditions $\phi(t = 0, x) = q(x)$ and $\dot{\phi}(t = 0, x) = p(x)$, we can find the solution for all later times:

$$f(x) + g(x) = q(x), \quad f'(x) - g'(x) = p(x).$$

so that

$$f(x) = \frac{1}{2}\int_0^x p(x')dx' + \frac{1}{2}q(x), \quad g(x) = -\frac{1}{2}\int_0^x p(x')dx' + \frac{1}{2}q(x)$$

The constant of integration is irrelevant as it will cancel when we add f and g together to get ϕ :

$$\phi(t, x) = \frac{1}{2}\left[q(x + t) + q(x - t)\right] + \frac{1}{2}\int_{x-t}^{x+t} p(x')dx'.$$

115.2 This solves the initial value problem for the wave equation.

Chapter 14

Laplace Equation

116 The Laplace equation

$$\frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi = 0$$

arises in electrostatics, Newton's theory of gravity, the theory of ideal fluids etc.

116.1 If $z = x + iy$ and $\partial = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right], \bar{\partial} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$ we can write the Laplace equation as

$$\partial\bar{\partial}\phi = 0.$$

117 The general solution of the Laplace equation is

$$\phi(x,y) = f(z) + g(\bar{z}).$$

117.1 Here, f is an analytic function and g is an anti-analytic function.

117.2 For example, if $\phi(x, y = 0) = q(x)$ and $\frac{\partial\phi}{\partial y}(x, y = 0) = p(x)$ are given,

$$f(x) = q(x) + p(x), \quad if'(x) - ig'(x) = p(x)$$

which allow us to solve for ϕ .

118 Fourier transforms give another way of solving this boundary value problem.

118.1 If we take the Fourier transform on the variable x ,

$$\phi(x,y) = \int e^{ikx}\tilde{\phi}(k,y)\frac{dk}{2\pi}$$

we will get

$$\frac{\partial^2}{\partial y^2}\tilde{\phi}(k,y) = k^2\tilde{\phi}(k,y)$$

Thus

$$\tilde{\phi}(k,y) = a(k)\cosh ky + b(k)\sinh ky.$$

Putting in the boundary conditions

$$a(k) = \int q(x)e^{-ikx}dx, \quad b(k) = \frac{1}{k}\int p(x)e^{-ikx}dx.$$

Now we can put back to get the answer needed.

Chapter 15

Initial Value problems

119 Many physical problems involve time evolution: given the state of the system at an initial time, a differential equation predicts its state at any future time.

119.1 For the wave equation, the initial state is given by $q(x) = \phi(t = 0, x)$ and $p(x) = \dot{\phi}(t = 0, x)$.

119.2 For the diffusion equation,

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} h$$

the initial state is determined by the value of the function $h(t = 0, x)$.

119.3 Certain properties of the state remain unchanged under time evolution. For the diffusion equation,

$$\frac{\partial}{\partial t} \int h(t, x) dx = 0.$$

For the wave equation,

$$\frac{\partial}{\partial t} \int [\dot{\phi}^2(t, x) + \phi'^2(t, x)] dx = 0.$$

These are examples of *conserved quantities*. The latter is the total energy of the wave.

119.4 Some other properties might be monotonic in time:

$$S(t) = -\int h(t,x)\log h(t,x)dx$$

has positive time derivative in the case of the heat equation:

$$\frac{\partial}{\partial t}S(t) \geq 0.$$

This has the physical meaning of entropy.

120 We can solve the initial value problem of the diffusion and wave equations by Fourier analysis.

120.1 Let $\tilde{h}(t,k) = \int e^{-ikx}h(t,x)dx$; it satisfies the ODE

$$\frac{\partial\tilde{h}(t,k)}{\partial t} = -k^2\tilde{h}(t,k).$$

The solution is

$$\tilde{h}(t,k) = \tilde{h}(0,k)e^{-k^2t} = e^{-k^2t}\int q(x')e^{-ikx'}dx'.$$

120.2 Thus

$$h(t,x) = \int q(x')K(t;x-x')dx'$$

where

$$K(t;x) = \int e^{-k^2t}e^{ikx}\frac{dk}{2\pi} = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{[4\pi t]}}.$$

Chapter 16

The Method of Characteristics

121 First order PDE can be solved by the method of characteristics.

121.1 Let $\phi : R^2 \to R$ be the unkown function. Given $v_1, v_2 : R^2 \to R$, consider the first order PDE

$$v_1 \frac{\partial \phi}{\partial x} + v_2 \frac{\partial \phi}{\partial y} = 0.$$

121.2 The change in ϕ due to an infinitesimal change in the position is

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy.$$

Hence, if we move to the point $(x + v_1(x,y)dt, y + v_2(x,y)dt)$ from (x,y) the function ϕ will not change: that is the meaning of the PDE. Here, dt is some infinitesimal quantity determining the size of the step; (v_1, v_2) determine the direction. At this new point, point we can take another small step determined by the new values of v_1 and v_2. Again, the value of ϕ will remain unchanged.

121.3 At each point (x, y) there is a vector $\mathbf{v}(x,y) = v_1(x,y)\mathbf{i} + v_2(x,y)\mathbf{j}$; along a curve which is tangential to this vector at each pint the function ϕ is a constant. This curve is determined by the ODE

$$\frac{dx(t)}{dt} = v_1(x(t), y(t)), \quad \frac{dy(t)}{dt} = v_2(x(t), y(t)).$$

121.4 Through each point there is one such curve; they cannot intersect each other. These are the *characteristic curves* of the PDE.

121.5 Example:

$$\frac{\partial\phi}{\partial x}-\frac{\partial\phi}{\partial y}=0.$$

The characteristic curves are $x(t)=x(0)+t, y(t)=y(0)-t$, so that $x+y=$ constant. Along these curves, the function ϕ is a constant.

121.6 This idea works in any dimension: given a vector field $(v_1,\cdots v_n)$, the solution to the equation

$$\sum_i v_i\frac{\partial\phi}{\partial x_i}=0$$

is constant along the curve determined by the ODE

$$\frac{dx_i(t)}{dt}=v_i(x(t)).$$

122 The solution of a linear first order PDE is determined by ODEs: determine the characetristic curve and then determine the variation along them.

122.1 The most general linear first order PDE is

$$\sum_i v_i\frac{\partial\phi}{\partial x_i}+f\phi+g=0.$$

The characteristic curves are determined by the part that has derivatives. Along each characteristic curve, we have the ODE

$$\frac{d\phi(x(t))}{dt}+f(x(t))\phi(x(t))+g(x(t))=0.$$

122.2 Example:

$$\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{dy} + 3\phi + x^2 = 0.$$

The characteristics are $x(t) = x(0) + t, y(t) = y(0)e^t$. Along them we have the ODE

$$\dot{\phi}(t) + 3\phi(t) + (x(0) + t)^2 = 0.$$

This can be solved and then putting in the formula for the characteristic curves, we get the solution of the PDE.