Chapter 1

Formal Power Series

1 Given a sequence of complex numbers $(a_0, a_1, a_2, \cdots)$, with only a finite number of non-zero entries, we have a polynomial with these numbers as coefficients:

$$a(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Although the notation makes it look like an infinite series, there are only a finite number of terms. The *order* of the polynomial is the largest n such that $a_n \neq 0$. We will also define the *degree* of the polynomial to be the *smallest* n such that $a_n \neq 0$.

2 Multiplication and addition of polynomials are really operations on their coefficients:

$$a + b(z) = \sum_n [a_n + b_n] z^n, \quad ab(z) = \sum_{n=0}^{\infty} \left[\sum_{\substack{k+l=n \\ k,l \geq 0}} a_k b_l \right] z^n$$

3 Under these operations the set of polynomials is a *commutative ring*, denoted by $C[z]$. In fact it is an *integral domain*; i.e., $ab = 0$ implies that either $a = 0$ or $b = 0$.

3.1 This can be proved by contradiction. Suppose a and b are both non-zero and $ab = 0$. Let k be the degree of a and l the degree of b

; then the ab has coefficient $a_k b_l$ for z^{k+l} which is non-zero; this cannot be, since $ab = 0$.

4 The *quotient field* of an integral domain A is the space of ordered pairs $(a, b), b \neq 0$ with the equivalence relation

$$(a, b) = (c, d) \iff ad - bc = 0.$$

The equivalence class is denoted by ab^{-1} or a/b . With the definitions $(a/b)+(c/d) = ((ad + bc/bd), \quad (a/b)(c/d) = (ac/bd) \quad (a/b)^{-1} = b/a$ this is indeed a field. The standard example is: the quotient field of the integers is the field of rational numbers.

5 The quotient field of the ring of polynomials is the field of *rational functions*.

6 The *derivative* of a polynomial is defined to be

$$Da(z) = \sum_{n=1} na_n z^{n-1} = \sum_{n=0}^{\infty}(n + 1)a_{n+1}z^n.$$

Clearly $D[ab] = [Da]b + a[Db]$.

7 A *formal power series* is a generalization of a polynomial where we allow $(a_0, a_1, \cdots)$ to be any sequence of complex numbers: with a possibly an infinite number of non-zero terms. It may no longer makes sense to evaluate the series $\sum_{n=0}^{\infty} a_n z^n$ for any number z . But since the operations on polynomials have a meaning directly in terms of the coeffiecients, we can define the sum, product and derivative of formal power series.

$$[a + b]_n = a_n + b_n, \quad [ab]_n = \sum_{\substack{k + l = n \\ k, l \geq 0}} a_k b_l.$$

The set of formal power series is a ring, indeed even an integral domain. (The proof is the same as above.) It is denoted by $C[[z]]$.

8 The derivative of a formal power series is

$$[Da]_n = (n + 1)a_{n+1}.$$

9 Although the series $\sum_{n=0} a_n z^n$ may not converge, it is still a useful notation to encode the sequence as it explains the motivation for the above definitions.

10 It is useful to have a notion of distance in the space of formal power series. Define the *degree* of a to be the smallest n such that $a_n \neq 0$; if $a = 0$ identically, we define its degree to be $+\infty$. Then define the distance between two series to be

$$d(a,b) = 2^{-\deg(a-b)}.$$

This is a metric:

$$d(a,b) \leq d(a,c) + d(c,b).$$

In fact $d(a,b) = \max\{d(a,c), d(c,b)\}$ so that is a non-archimedean metric or *ultrametric*. The space of formal power series is the completion of the space of polynomials under this metric.

10.1 To see $d(a,b) = \max\{d(a,c), d(c,b)\}$, suppose $d(a,c) = 2^{-n_1}, d(c,b) = 2^{-n_2}$ with $n_1 \geq n_2$. Then c agrees with a upto n_1 , and with b upto n_2 . So a agrees with b upto n_2 .

11 The *reciprocal* of a formal series a of degree zero (i.e., $a_0 \neq 0$) is the unique power series $\frac{1}{a}(z)$ such that $a(z)\frac{1}{a}(z) = 1$. Its coefficients are determined by solving the linear system

$$\begin{pmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ . & . & . & \cdots \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ . \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \end{pmatrix}.$$

12 The quotient field of the ring of formal power series is the field of sequences $a : Z \to C$ labelled by the integers such that, there exists a d with $a_n = 0$ whenever $n < d$. (This is the space of *formal Laurent series*.) To see this, notice that a/b can be identified with $z^{-k}a(z)\frac{1}{z^{-k}b(z)}$ where k is the degree of b .

13 The translation of a series by a constant, $[T_\alpha a]_n = \sum_{m=0}^{\infty} a_{m+n} \binom{m+n}{n} \alpha^m$ is defined whenever this infinite series converges; e.g., when $a(z)$ is an entire function.

14 The *substitution* of b into a (also called the *composition*) $a\circ b$ is defined if either $b_0=0$ or if a is a polynomial:

$$[a\circ b]_n=\sum_{k=0}^{\infty}a_k\sum_{\substack{l_1+\cdots l_k=n\\ l_1,l_2\cdots\geq 0}}b_{l_1}b_{l_2}\cdots b_{l_k}.$$

The point is that, for each n there are only a finite number of such l 's so that the series on the rhs is really a finite series.

14.1 It is actually sufficient that a is entire for $a\circ b$ to exist; i.e., that $\sum_{n=0}a_nz^n$ converge for any z . For, we can split $b(z)=b_0+\tilde b(z)$ with $\tilde b_0=0$; then $a\circ b=T_{b_0}a\circ\tilde b$ is defined for a entire. The composition of power series is not always defined. Consider for example,

$$a(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\quad b(z)=1.$$

We will mostly use composition $a\circ b$ when $b_0=0$.

15 The set of formal power series

$$\mathcal{G}=\{\sum_0^{\infty}\phi_nz^n|\phi_0=0;\phi_1\neq 0\}$$

is a group under composition: the group of *formal diffeomorphisms* or *formeomorphisms*. The composition is

$$[\psi\circ\phi]_n=\sum_{k=1}^{n}\psi_k\sum_{l_1+l_2\cdots+l_k=n}\phi_{l_1}\cdots\phi_{l_k}$$

The inverse of ϕ (say ψ) is determined by the recursion relations

$$\psi_1\phi_1=1,\quad \psi_n=-\frac{1}{\phi_1^n}\sum_{k=1}^{n-1}\psi_k\sum_{l_1+\cdots l_k=n}\phi_{l_1}\cdots\phi_{l_k}.$$

15.1 The group $\mathcal{G}$ has a representation (the *defining representation*) on the space of formal power series:

$$[\phi^*a]_n=[a\circ\phi]_n=\sum_{k=1}^{n}a_k\sum_{l_1+l_2+\cdots+l_k=n}\phi_{l_1}\cdots\phi_{l_k}.$$

Just as a formal power series are a generalization of the notion of a polynomial or smooth function, a formeomorphism is a generalization of the notion of a change of independent variable or diffeomorphism.

15.2 $\mathcal{G}$ is a topological group with respect to the ultrametric topology described above. The subset of polynomials with $\phi_0 = 0, \phi_1 \neq 0$ is not a group, only a monoid; the inverse of a polynomial is not usually a polynomial.

16 An infinitesimal formeomorphism (*formal vector field*) is a derivation of the algebra of formal power series. They form the Lie algebra $\underline{\mathcal{G}}$ of the group of formeomorphisms; it is a topological Lie algebra with the ultrametric d . A basis is

$$L_n = x^{n+1} D, n = 1, 2, \cdots$$

satisfying the commutation relations

$$[L_m, L_n] = (n - m) L_{m+n}.$$

This is also called the *Virasoro* algebra by some physicists and the *Witt* algebra by some mathematicians.

17 The left action of this Lie algebra on the group $\mathcal{G}$ is given by

$$[L_n \phi]_k = \sum_{l_1 + l_2 \cdots l_n = k} \phi_{l_1} \cdots \phi_{l_n}$$

or equivalently $L_n \phi(x) = \phi(x)^n$. The right action is given by

$$R_n \phi(x) = x^{n+1} D \phi(x).$$