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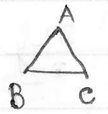
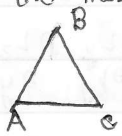
PROBLEM SET 1 — ARNAB KAR

1. The group S_3 ~~have~~ has $3! = 6$ elements in it.
To find the matrix representation we consider an equilateral triangle and all the possible rotations and reflections which will give rise to the permutation of its vertices taken in an order (say anticlockwise).
First of all we have identity element corresponding to this configuration -



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow ABC$$

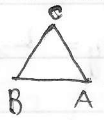
We then do a rotation by 120° and the matrix doing so becomes the next element of the representation.



$$\begin{pmatrix} \cos 120^\circ & \sin 120^\circ \\ -\sin 120^\circ & \cos 120^\circ \end{pmatrix} = \begin{pmatrix} -\cos 60^\circ & \sin 60^\circ \\ -\sin 60^\circ & -\cos 60^\circ \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$\rightarrow$ BAC

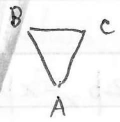
We can do a rotation by 240° and get one more configuration -



$$\begin{pmatrix} \cos 240^\circ & \sin 240^\circ \\ -\sin 240^\circ & \cos 240^\circ \end{pmatrix} = \begin{pmatrix} -\cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

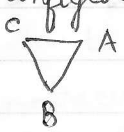
$\rightarrow$ CBA

We can now look into the reflection of the original configuration which will be brought about by the parity matrix for 2D -

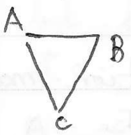


$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow ACB$$

We can now do rotations by 120°, 240° like before to this configuration to give the two more elements.



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \rightarrow BAC$$



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \rightarrow CBA$$

Thus the following matrices will form a representation of S_3 -

$$\underset{ABC}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{BAC}{\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}, \underset{CBA}{\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}, \underset{ACB}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, \underset{BAC}{\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}}, \underset{CBA}{\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}}$$

2. $L_x = r_y p_z - r_z p_y = y p_z - z p_y$

$L_y = r_z p_x - r_x p_z = z p_x - x p_z$

$L_z = r_x p_y - r_y p_x = x p_y - y p_x$

$$\{L_x, L_y\} = \frac{\partial L_x}{\partial x}\frac{\partial L_y}{\partial p_x} - \frac{\partial L_x}{\partial p_x}\frac{\partial L_y}{\partial x} + \frac{\partial L_x}{\partial y}\frac{\partial L_y}{\partial p_y} - \frac{\partial L_x}{\partial p_y}\frac{\partial L_y}{\partial y} + \frac{\partial L_x}{\partial z}\frac{\partial L_y}{\partial p_z} - \frac{\partial L_x}{\partial p_z}\frac{\partial L_y}{\partial z}$$

$$= 0 - 0 + 0 - 0 + (-p_y)(-x) - y p_x = x p_y - y p_x = L_z$$

$$\{L_y, L_z\} = \frac{\partial L_y}{\partial x}\frac{\partial L_z}{\partial p_x} - \frac{\partial L_y}{\partial p_x}\frac{\partial L_z}{\partial x} + \frac{\partial L_y}{\partial y}\frac{\partial L_z}{\partial p_y} - \frac{\partial L_y}{\partial p_y}\frac{\partial L_z}{\partial y} + \frac{\partial L_y}{\partial z}\frac{\partial L_z}{\partial p_z} - \frac{\partial L_y}{\partial p_z}\frac{\partial L_z}{\partial z}$$

$$= (-p_z)(-y) - z p_y + 0 - 0 + 0 - 0 = y p_z - z p_y = L_x$$

$$\{L_z, L_x\} = \frac{\partial L_z}{\partial x}\frac{\partial L_x}{\partial p_x} - \frac{\partial L_z}{\partial p_x}\frac{\partial L_x}{\partial x} + \frac{\partial L_z}{\partial y}\frac{\partial L_x}{\partial p_y} - \frac{\partial L_z}{\partial p_y}\frac{\partial L_x}{\partial y} + \frac{\partial L_z}{\partial z}\frac{\partial L_x}{\partial p_z} - \frac{\partial L_z}{\partial p_z}\frac{\partial L_x}{\partial z}$$

$$= 0 - 0 + (-p_x)(-z) - (x)(p_z) + 0 - 0 = z p_x - x p_z = L_y$$

Thus we end up with the following relations -

$\{L_x, L_y\} = L_z$, $\{L_z, L_x\} = L_y$, $\{L_y, L_z\} = L_x$

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k$$

These have similar Lie algebra as the rotation matrices in 3D which are antisymmetric. i.e. $A^T = -A$. Hence such a matrix is represented by $(9-3)/2 = 3$ independent components.

$$\begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix} = A_{12}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + A_{13}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + A_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= A_{12} S_{12} + A_{13} S_{13} + A_{23} S_{23}$$

$$S_{12} S_{13} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{13} S_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_{12} S_{13} - S_{13} S_{12} = S_{23} = [S_{12}, S_{13}]$$

Similarly it can be shown that -

$$S_{23} S_{12} - S_{12} S_{23} = S_{13} = [S_{23}, S_{12}]$$
$$S_{13} S_{23} - S_{23} S_{13} = S_{12} = [S_{13}, S_{23}]$$

$$S_{12} S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[S_{12}, S_{12}] = [S_{23}, S_{23}] = [S_{13}, S_{13}] = 0$$

Thus a faithful representation of the Poisson brackets will be under the injective mapping -

$$L_x \to S_{12}, \quad L_y \to S_{13}, \quad L_z \to S_{23}$$

Now to find a correspondence with anti hermitian ~~two~~ 2x2 matrices we ~~can~~ use the Pauli matrices which are hermitian but have similar commutation relations.

$$[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = i\varepsilon_{ijk}\sigma_k - i\varepsilon_{jik}\sigma_k = 2i\varepsilon_{ijk}\sigma_k$$

We can take care of the factor 2 by dividing all matrices by 2.

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\varepsilon_{ijk}\frac{\sigma_k}{2}$$

Now we need to make them anti hermitian which we can by multiplying each of the matrices by i.

$[\frac{i}{2}\sigma_k, \frac{i}{2}\sigma_j] = (i)^2 \varepsilon_{kjl} \frac{i\sigma_l}{2} = -\varepsilon_{kjl} \left(\frac{i\sigma_l}{2}\right)$

Thus we can have $A = \frac{i\sigma_1}{2}$, $B = \frac{i\sigma_2}{2}$, $C = \frac{i\sigma_3}{2}$

$[A, B] = [B, A] = \left(\frac{i}{2}\right)^2 [\sigma_2, \sigma_1] = +\frac{1}{4} \times 2i\sigma_3 = \frac{i\sigma_2}{2} = C$

$[C, B] = A, \quad [A, C] = B$

$[A, A] = [B, B] = [C, C] = 0 \qquad (\because \sigma_i^2 = I)$

Thus the Poisson brackets of L_i's could be represented by these matrices on a one to one mapping -

$L_x \to C, \ L_y \to B, \ L_z \to A$

3. Let us denote the group of 2x2 Traceless matrices by the group T. The operation associated with this group is addition.

If $t_1, t_2 \in T$. i.e. $Tr(t_1) = 0$, $Tr(t_2) = 0$.

$Tr(t_1 + t_2) = Tr(t_1) + Tr(t_2) = 0$.

We now want to find an isomorphism between T and $sl(2, \mathbb{R})$ which is given by the following map -

$\exp : T \to sl(2, \mathbb{R})$

To show that this mapping is a homomorphism first, we use the following property of matrices.

$\det(e^A) = e^{Tr A}$ (A being a matrix)

If $t_1, t_2 \in T$ and are mapped to $d_1, d_2 \in sl(2, \mathbb{R})$ under this mapping, then -

$\exp(t_1 + t_2) = \exp(t_1) \cdot \exp(t_2) = d_1 \cdot d_2$

To show that $d_1, d_2 \in sl(2, \mathbb{R})$ it should have determinant one.

$\det(d_1 \cdot d_2) = \det(d_1) \cdot \det(d_2)$

$= \det(e^{t_1}) \cdot \det(e^{t_2})$

$= e^{Tr t_1} \, e^{Tr t_2} = e^{Tr t_1 + Tr t_2} = e^0 = 1$

To show that this homomorphism is an isomorphism, the kernel should have just one element.

$\ker(\exp) = \{ t \in T \mid \exp(T) = \mathbb{I}_{2\times 2} \}$

The only way we can get the 2×2 identity matrix is when the traceless matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence the kernel consists of only one element and the homomorphism becomes injective as well.

Since all the matrices of det 1 in $sl(2, \mathbb{R})$ are generated from traceless matrices, we can look into the basis of all the traceless matrices and generate all the elements in $sl(2, \mathbb{R})$.

The basis choice for traceless matrices could be –

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We now look into the Lie algebra associated with it.

$$AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2B$$

$\cancel{[B, A] = -2B}$ $\quad [A, B] = -2B.$

$$AC - CA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & +2 \\ 0 & 0 \end{pmatrix} = 2C$$

$[A, C] = 2C$

$$CB - BC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A$$

$[C, B] = A$

Functions of position and momentum which may have Poisson bracket relation similar to this Lie algebra are J_+, J_-, $2iJ_z$.

$J_+ = \dfrac{J_x + iJ_y}{2}$, $\quad J_- = \dfrac{J_x - iJ_y}{2}$, $\quad J_K = \varepsilon_{ijK}\, x_i\, p_j$

x, p being position and momentum.

We know that $\{J_i, J_j\} = \varepsilon_{ijk} J_k$

$$\{J_+, 2J_z\} = \{\tfrac{J_x + iJ_y}{2}, 2J_z\} = \{J_x, 2J_z\} + i\{J_y, 2J_z\}$$
$$= -2J_y + 2J_x = 2(J_x - J_y)$$

$$\{J_-, 2J_z\} = \{J_x - J_y, 2J_z\} = \{J_x, 2J_z\} - \{J_y, 2J_z\}$$
$$= -2J_y - 2J_x = -2(J_x + J_y)$$

$$\{J_+, J_-\} = \{J_x + J_y, J_x - J_y\} = \{J_x, J_x\} + \{J_y, J_x\}$$
$$- \{J_x, J_y\} - \{J_y, J_y\}$$
$$= 0 - J_z - J_z - 0 = -2J_z$$

We can rewrite the above as -

$$\{2J_z, J_+\} = -2J_-$$
$$\{2J_z, J_-\} = 2J_+$$
$$\{J_-, J_+\} = 2J_z$$

Thus we see that the Poisson brackets for these have similar structure as Lie algebra of $sl(2, \mathbb{R})$.
We can map the basis matrices of (A, B, C) with $(J_+, J_-, 2J_z)$ and find an isomorphism.

$A \to 2J_z, \quad B \to J_+, \quad C \to J_-$

$$\{J_+, 2iJ_z\} = \{\frac{J_x + iJ_y}{2}, 2iJ_z\}$$

$$= \{\frac{J_x}{2}, 2iJ_z\} + \frac{i}{2}\{J_y, 2iJ_z\} = -iJ_y - J_x = -2J_+$$

$$\{J_-, 2iJ_z\} = \{\frac{J_x - iJ_y}{2}, 2iJ_z\}$$

$$= i\{J_x, J_z\} + \{J_y, J_z\} = -iJ_y + J_x = 2J_-$$

$$\{J_+, J_-\} = \{\frac{J_x + iJ_y}{2}, \frac{J_x - iJ_y}{2}\}$$

We now look for functions of position and momentum which satisfy similar relations under Poisson brackets.

$$\{xp, \frac{x^2}{2}\} = \frac{\partial(xp)}{\partial x}\frac{\partial(x^2/2)}{\partial p} - \frac{\partial(xp)}{\partial p}\frac{\partial(x^2/2)}{\partial x} = 0 - x.x = -2.\frac{x^2}{2}$$

$$\{xp, -\frac{p^2}{2}\} = \frac{\partial(xp)}{\partial x}\frac{\partial(-p^2/2)}{\partial p} - \frac{\partial(xp)}{\partial p}\frac{\partial(-p^2/2)}{\partial x} = -p.\frac{2p}{2} = 2\left(-\frac{p^2}{2}\right)$$

$$\{\frac{x^2}{2}, -\frac{p^2}{2}\} = \frac{\partial(x^2/2)}{\partial x}\frac{\partial(-p^2/2)}{\partial p} - \frac{\partial(x^2/2)}{\partial p}\frac{\partial(-p^2/2)}{\partial x} = \frac{2x}{2}\left(\frac{-2p}{2}\right) - 0 = -xp$$

$$\{-\frac{p^2}{2}, \frac{x^2}{2}\} = xp$$

So we see that these have similar Lie algebra as that $sl(2,\mathbb{R})$.
We can now map the matrices to these functions now and get an isomorphism.

$$A \to xp, \quad B \to \frac{x^2}{2}, \quad C \to -\frac{p^2}{2}$$

To show that $\det(e^A) = e^{\operatorname{Tr} A}$

For a matrix A with non zero determinant it can be diagonalised and represented as $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, with λ_1, λ_2 being eigen values.

$$e^A = \mathbb{1} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$= \mathbb{1} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} + \dots$$

$$= \begin{pmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2!}+\dots & 0 \\ 0 & 1+\frac{\lambda_1}{1}+\frac{\lambda_2^2}{2!}+\dots \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

$$\det(e^A) = e^{\lambda_1} e^{\lambda_2} = e^{\lambda_1+\lambda_2} = e^{\operatorname{Tr} A}$$

$$\operatorname{Tr} A = \lambda_1 + \lambda_2$$