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PROBLEM SET 3 ARNAB KAR

2. $(\vec{J}+\vec{J}')^2 = J^2 + J'^2 + 2\,\vec{J}\cdot\vec{J}'$

$= J^2 + J'^2 + 2J_xJ_x' + 2J_yJ_y' + 2J_zJ_z'$

$J_x = \frac{\sigma_1}{2}$, $J_y = \frac{\sigma_2}{2}$, $J_z = \frac{\sigma_3}{2}$

$J_x' = \frac{\sigma_1'}{2}$, $J_y' = \frac{\sigma_2'}{2}$, $J_z' = \frac{\sigma_3'}{2}$

$J_x\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}$ $\quad J_x|\uparrow\rangle = \frac{1}{2}|\downarrow\rangle$

$J_y\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\i\end{pmatrix}$ $\quad J_y|\uparrow\rangle = \frac{i}{2}|\downarrow\rangle$

$J_z\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}$ $\quad J_z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle$

$J_x\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}$ $\quad J_x|\downarrow\rangle = \frac{1}{2}|\uparrow\rangle$

$J_y\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-i\\0\end{pmatrix}$ $\quad J_y|\downarrow\rangle = -\frac{i}{2}|\uparrow\rangle$

$J_z\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}$ $\quad J_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$

Similarly for the other set of $\vec{J}'$.

using these relations along with $\sigma_i^2 = 1 \Rightarrow J_x^2 = J_y^2 = J_z^2 = \frac{1}{4}$.

$(\vec{J}+\vec{J}')^2|\uparrow\uparrow\rangle = (J^2 + J'^2 + 2J_xJ_x' + 2J_yJ_y' + 2J_zJ_z')|\uparrow\uparrow\rangle$

$= \left[\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) + 2\cdot\frac{1}{2}\cdot\frac{1}{2}\right]|\uparrow\uparrow\rangle$

$+2\left[\frac{1}{2}\cdot\frac{1}{2} + \frac{i}{2}\cdot\frac{i}{2}\right]|\downarrow\downarrow\rangle$

$= 2|\uparrow\uparrow\rangle + 0|\downarrow\downarrow\rangle = 2|\uparrow\uparrow\rangle$

(The last two terms come from x, y components)

(Using $J_x^2|\uparrow\rangle = J_x\frac{1}{2}|\downarrow\rangle = \frac{1}{2}J_x|\downarrow\rangle = \frac{1}{4}|\uparrow\rangle$, & $J_y^2|\uparrow\rangle = \frac{i}{2}J_y|\downarrow\rangle = \left(-\frac{i^2}{4}\right)|\uparrow\rangle$)

$(\vec{J}+\vec{J}')^2|\downarrow\downarrow\rangle = (J^2 + \vec{J}'^2 + 2J_xJ_x' + 2J_yJ_y' + 2J_zJ_z')|\downarrow\downarrow\rangle$

$= \left[\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) + 2\cdot\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right]|\downarrow\downarrow\rangle$

$+2\left[\frac{1}{2}\cdot\frac{1}{2} + \left(\frac{-i}{2}\right)\left(-\frac{i}{2}\right)\right]|\uparrow\uparrow\rangle$

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$$= 2\,|\downarrow\downarrow\rangle + 0\,|\uparrow\uparrow\rangle = 2\,|\downarrow\downarrow\rangle$$

$$(\vec{J} + \vec{J}')^2 \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$= \left(J^2 + J'^2 + 2J_x J_x' + 2J_y J_y' + 2J_z J_z'\right)\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$= \left[2\left(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}\right) + 2\left(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}\right) + 2\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) + 2\left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\right]$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) + \frac{2}{\sqrt{2}}\left[\tfrac{1}{2}\cdot\tfrac{1}{2}\right]|\downarrow\uparrow\rangle + \frac{2}{\sqrt{2}}\left[\tfrac{1}{2}\cdot\tfrac{1}{2}\right]|\uparrow\downarrow\rangle$$

$$+ \frac{2}{\sqrt{2}}\left[\tfrac{i}{2}\left(-\tfrac{i}{2}\right)\right]|\downarrow\uparrow\rangle + \frac{2}{\sqrt{2}}\left[-\tfrac{i}{2}\cdot\tfrac{i}{2}\right]|\uparrow\downarrow\rangle$$

$$= \frac{2}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) + \left(\tfrac{1}{2}\right)\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$+ \left(\tfrac{1}{2}\right)\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big[\; J^2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) + J'^2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$+ 2J_z J_z'\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) + 2J_x J_x'\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$+ 2J_y J_y'\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\Big]$$

$$= \frac{1}{\sqrt{2}}\left[\left(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}\right)|\uparrow\downarrow\rangle + \left(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}\right)|\downarrow\uparrow\rangle + \left(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}\right)|\uparrow\downarrow\rangle\right.$$

$$+ \left(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}\right)|\downarrow\uparrow\rangle + 2\cdot\tfrac{1}{2}\left(-\tfrac{1}{2}\right)|\uparrow\downarrow\rangle + 2\left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)|\uparrow\downarrow\rangle$$

$$+ 2\,\tfrac{1}{2}\cdot\tfrac{1}{2}\,|\downarrow\uparrow\rangle + 2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\,|\uparrow\downarrow\rangle + 2\cdot\tfrac{i}{2}\left(-\tfrac{i}{2}\right)|\downarrow\uparrow\rangle$$

$$\left. + 2\left(-\tfrac{i}{2}\right)\cdot\left(\tfrac{i}{2}\right)|\uparrow\downarrow\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{3}{4}\times 2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) - \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\right.$$

You can do higher spins than 1 too. in Prob 1. But ok.

$$+ \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) + \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\Big]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{3}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right](|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$= 2\,\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$(\vec{J} + \vec{J}')^2 \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$= (J^2 + J'^2 + 2J_z J_z' + 2J_x J_x' + 2J_y J_y')\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$= \frac{1}{\sqrt{2}}\Big[\frac{3}{4}|\uparrow\downarrow\rangle - \frac{3}{4}|\downarrow\uparrow\rangle + \frac{3}{4}|\uparrow\downarrow\rangle - \frac{3}{4}|\downarrow\uparrow\rangle$$

$$+ 2\,\frac{1}{4}|\uparrow\downarrow\rangle - 2.\frac{1}{4}|\downarrow\uparrow\rangle + 2.\frac{1}{4}|\downarrow\uparrow\rangle - \frac{2}{4}|\uparrow\downarrow\rangle$$

$$+ 2\left(\frac{i}{2}\right)\left(\frac{-i}{2}\right)|\downarrow\uparrow\rangle - 2\left(\frac{-i}{2}\right)\left(\frac{i}{2}\right)|\uparrow\downarrow\rangle\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[\frac{3}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) - \frac{1}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) - \frac{1}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) - \frac{1}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\Big]$$

$$= 0\left(\frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]\right)$$

Thus singlet and triplet states have 0 and 2 as eigen values for $(\vec{J} + \vec{J}')^2$.

3. A symmetric tensor's independent components is same as the no. of ways r particles can be placed in n energy levels with more than one particle occupying one level (bosonic system).

Let us arrange the particles now in the energy levels.

$$\left.\begin{array}{l} ||\ldots|| \rightarrow x_n \\ ||\ldots|| \\ \vdots \\ ||\ldots|| \rightarrow x_1 \end{array}\right\} n \text{ levels}$$

x_i denotes the no. of particles in ith level.

Since the total no. of particles is constant, we have–

$x_1 + \dots + x_n = r.$

The no. of ways in which they can be put is–

$$S = \frac{(n-1+r)!}{r!\, n-1!}$$

It can be seen as if rearranging $(n-1)$ "+" signs are r sticks in this config.

$$\underbrace{||\dots||}_{x_1} \text{"+"} ||\dots|| + \dots \text{"+"} \underbrace{||\dots||}_{x_n}$$

To derive the above formula we can think of ~~putti~~ it as– We have $(n-1+r)$ objects to arrange which can be arranged in $(n-1+r)!$ ways. Now assign the position to sticks and "+" is S ways. Now the ~~sticks~~ "+" can be rearranged among them in $(r)!$ ways and sticks can rearrange in $(n-1)!$ ways.

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$$S = \frac{(n-1+r)!}{r!\, n-1!} = \frac{(n+r-1)(n+r-2)\dots(n-1+r-n+1)}{r!}$$

$$= \frac{(n-1+r)(n-2+r)\dots(n)}{r!}$$

Similarly an antisymmetric tensor dimension can be viewed as the no. of ways r particles can be placed in n levels without any repetations like fermionic systems.
The first ~~energy~~ particle ~~level~~ can be fitted into any of one of n levels leaving $(n-1)$ levels for second particle and consequently $(n-r+1)$ levels left for last particle. They can they rearrange among them in $r!$ ways representing some state.
Thus no. of states possible are–

$$\frac{(n)(n-1)\dots(n-(r-1))}{r!}$$

1. $j = \frac{1}{2}$ $\hbar = 1$ (say)

$L_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $L_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $L_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

For Pauli matrices, $\sigma_i^2 = \mathbb{1}$

$$M = z_1 L_1^2 + z_2 L_2^2 + z_3 L_3^2$$

$$= \frac{z_1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{z_2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{z_3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{(z_1+z_2+z_3)}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det M = \left(\frac{z_1+z_2+z_3}{4}\right)^2 = 0.$$

$$z_i = \frac{1}{2A_i} - \frac{E}{\frac{1}{2}(1+\frac{1}{2})} = \frac{1}{2A_i} - \frac{4E}{3}$$

$$z_1 + z_2 + z_3 = 0$$

$$\frac{1}{2A_1} + \frac{1}{2A_2} + \frac{1}{2A_3} - 3\times\frac{4E}{3} = 0$$

$$\therefore E = \frac{1}{8}\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3}\right)$$

For $j=1$.

$$L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$L_+ |1,1\rangle = 0$ $L_- |1,1\rangle = \sqrt{2}\,|1,0\rangle$

$L_+ |1,0\rangle = \sqrt{2}\,|1,1\rangle$ $L_- |1,0\rangle = \sqrt{2}\,|1,-1\rangle$

$L_+ |1,-1\rangle = \sqrt{2}\,|1,0\rangle$ $L_- |1,-1\rangle = 0$

$L_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$ $L_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$

$$L_1 = \frac{L_+ + L_-}{2} = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$L_2 = \frac{L_+ - L_-}{2i} = \frac{1}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

$$L_1^2 = \frac{1}{4}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$L_2^2 = L_2^* L_2 = \frac{1}{4}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

$$M = z_1 L_1^2 + z_2 L_2^2 + z_3 L_3^2$$

$$= \frac{1}{4}\begin{pmatrix} 2z_1 & 0 & 2z_1 \\ 0 & 4z_1 & 0 \\ 2z_1 & 0 & 2z_1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2z_2 & 0 & -2z_2 \\ 0 & 4z_2 & 0 \\ -2z_2 & 0 & 2z_2 \end{pmatrix} + z_3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{z_1+z_2}{2} + z_3 & 0 & \frac{z_1-z_2}{2} \\ 0 & z_1+z_2 & 0 \\ \frac{z_1-z_2}{2} & 0 & \frac{z_1+z_2}{2} + z_3 \end{pmatrix}$$

$R_1 \to R_1 + R_3$

$$= \begin{pmatrix} z_1 + z_3 & 0 & z_1+z_3 \\ 0 & z_1+z_2 & 0 \\ \frac{z_1-z_2}{2} & 0 & \frac{z_1+z_2}{2} + z_3 \end{pmatrix}$$

$C_1 \to C_1 + C_3$

$$= \begin{pmatrix} 2(z_1+z_3) & 0 & z_1+z_3 \\ 0 & z_1+z_2 & 0 \\ z_1+z_3 & 0 & \frac{z_1+z_2}{2} + z_3 \end{pmatrix}$$

$$\det M = 2(z_1+z_3)(z_1+z_2)\left(\frac{z_1+z_2}{2} + z_3\right) - (z_1+z_3)(z_1+z_2)(z_1+z_3)$$

$$= (z_1+z_3)(z_1+z_2)(z_1+z_2+2z_3 - z_1 - z_3)$$

$$= (z_1+z_2)(z_1+z_3)(z_2+z_3)$$

$\therefore z_1 = -z_2, \quad z_3 = -z_1, \quad z_2 = -z_3.$

$$\frac{1}{2A_1} - \frac{E}{2} = -\frac{1}{2A_2} + \frac{E}{2}$$

$$\therefore E = \frac{1}{2A_1} + \frac{1}{2A_2} \quad \text{or} \quad E = \frac{1}{2A_3} + \frac{1}{2A_1} \quad \text{or} \quad E = \frac{1}{2A_2} + \frac{1}{2A_3}$$

$j = \frac{3}{2}$

$$J_3 = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

$J_+ \left|\frac{3}{2}, \frac{3}{2}\right\rangle = 0$ $\qquad$ $J_- \left|\frac{3}{2}, \frac{3}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2}, \frac{1}{2}\right\rangle$

$J_+ \left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2}, \frac{3}{2}\right\rangle$ $\qquad$ $J_- \left|\frac{3}{2}, \frac{1}{2}\right\rangle = 2\left|\frac{3}{2}, -\frac{1}{2}\right\rangle$

$J_+ \left|\frac{3}{2}, -\frac{1}{2}\right\rangle = 2\left|\frac{3}{2}, \frac{1}{2}\right\rangle$ $\qquad$ $J_- \left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2}, -\frac{3}{2}\right\rangle$

$J_+ \left|\frac{3}{2}, -\frac{3}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle$ $\qquad$ $J_- \left|\frac{3}{2}, -\frac{3}{2}\right\rangle = 0$

$$J_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$J_1 = \frac{J_+ + J_-}{2} = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$J_2 = \frac{J_+ - J_-}{2i} = \frac{1}{2i}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

$$J_1^2 = \frac{1}{4}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 0 & 3 & 0 \\ 0 & 7 & 0 & 3 \\ 3 & 0 & 7 & 0 \\ 0 & 3 & 0 & 3 \end{pmatrix}$$

$$J_2^2 = \frac{1}{4}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 0 & -3 & 0 \\ 0 & 7 & 0 & -3 \\ -3 & 0 & 7 & 0 \\ 0 & -3 & 0 & 3 \end{pmatrix}$$

$$J_3^{\,2} = \begin{pmatrix} \frac{9}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{9}{4} \end{pmatrix}$$

$$M = \frac{z_1}{4}\begin{pmatrix} 3 & 0 & 3 & 0 \\ 0 & 7 & 0 & 3 \\ 3 & 0 & 7 & 0 \\ 0 & 3 & 0 & 3 \end{pmatrix} + \frac{z_2}{4}\begin{pmatrix} 3 & 0 & -3 & 0 \\ 0 & 7 & 0 & -3 \\ -3 & 0 & 7 & 0 \\ 0 & -3 & 0 & 3 \end{pmatrix} + \frac{z_3}{4}\begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3z_1+3z_2+9z_3 & 0 & 3z_1-3z_2 & 0 \\ 0 & 7(z_1+z_2)+z_3 & 0 & 3(z_1-z_2) \\ 3z_1-3z_2 & 0 & 7(z_1+z_2)+z_3 & 0 \\ 0 & 3(z_1-z_2) & 0 & 3(z_1+z_2)+9z_3 \end{bmatrix}$$

$C_1 \to C_1 + C_3$

$$= \frac{1}{4}\begin{bmatrix} 6z_1+9z_3 & 0 & 3(z_1-z_2) & 0 \\ 0 & 7(z_1+z_2)+z_3 & 0 & 3(z_1-z_2) \\ 10z_1+4z_2+z_3 & 0 & 7(z_1+z_2)+z_3 & 0 \\ 0 & 3(z_1-z_2) & 0 & 3(z_1+z_2)+9z_3 \end{bmatrix}$$

$C_2 \to C_2 + C_3$

$$= \frac{1}{4}\begin{bmatrix} 6z_1+9z_3 & 3(z_1-z_2) & 3(z_1-z_2) & 0 \\ 0 & 7(z_1+z_2)+z_3 & 0 & 3(z_1-z_2) \\ 10z_1+4z_2+z_3 & 7(z_1+z_2)+z_3 & 7(z_1+z_2)+z_3 & 0 \\ 0 & 3(z_1-z_2) & 0 & 3(z_1+z_2)+9z_3 \end{bmatrix}$$

$a = 3z_1+3z_2+9z_3, \quad b = 3(z_1-z_2), \quad c = 7(z_1+z_2)+z_3$

$$M = \frac{1}{4}\begin{bmatrix} a & 0 & b & 0 \\ 0 & c & 0 & b \\ b & 0 & c & 0 \\ 0 & b & 0 & a \end{bmatrix}$$

$$\det M = a \det\begin{pmatrix} c & 0 & b \\ 0 & c & 0 \\ b & a & a \end{pmatrix} + b \det\begin{pmatrix} 0 & c & b \\ b & 0 & 0 \\ 0 & b & a \end{pmatrix}$$

$= a\left(c(ac) + b(-bc)\right) + b(b^3)$

$= a(ac^2 - b^2c) + b^4 = ac(ac-b^2) + b^4$

$= a^2c^2 + 2acb^2 + b^4 - acb^2$

$= (ac+b^2)^2 - acb^2$

$= (ac + b^2 + b\sqrt{ac})(ac + b^2 - \sqrt{ac}\, b)$

$\therefore ac + b^2 = -b\sqrt{ac}$ OR $ac + b^2 = b\sqrt{ac}$

We can put back z_1, z_2, z_3 in terms of a, b, c.

For a arbitrary spin j, we have -

$$L_3 = \begin{pmatrix} j & & & & 0 \\ & j-1 & & & \\ & & \ddots & & \\ & & & -j+1 & \\ 0 & & & & -j \end{pmatrix}$$

$$L_1 = \frac{1}{2}\begin{pmatrix} 0 & b_j & & & \\ b_j & 0 & b_{j-1} & & \\ & b_{j-1} & 0 & & \\ & & \ddots & & b_{-j+1} \\ & & & b_{-j+1} & 0 \end{pmatrix} \qquad b_i = \sqrt{(j+i)(j+1-i)}$$

$$L_2 = \frac{1}{2}\begin{pmatrix} 0 & -ib_j & & & \\ ib_j & 0 & -ib_{j-1} & & \\ & ib_{j-1} & 0 & & \\ & & \ddots & & -ib_{-j+1} \\ & & & ib_{-j+1} & 0 \end{pmatrix}$$

$$L_+ = \frac{L_1 + iL_2}{2} = \frac{1}{2}\begin{pmatrix} 0 & b_j & & & \\ 0 & 0 & b_{j-1} & & \\ & & \ddots & & \\ & 0 & & 0 & b_{-j+1} \\ & & & & 0 \end{pmatrix}$$

$$L_- = \frac{L_1 - iL_2}{2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & & & 0 \\ b_j & 0 & & & \\ & b_{j-1} & \ddots & & \\ & & & 0 & \\ & & & b_{-j+1} & 0 \end{pmatrix}$$

$\cancel{L_1^2 + L_2^2} = \cancel{L_-L_+} + L_3 =$