

1 Introduction

1.1 Most problems in physics cannot be solved exactly.

The few that that can be are solved by exploiting symmetries. The original example is the Kepler problem of a planet moving around the Sun under the influence of gravity. The symmetry under rotations allows this problem to be reduced to a one dimensional problem. Another example is the hydrogen atom where again the Schrodinger equation can be reduced to a one dimensional equation using spherical symmetry.

1.2 The fundamental laws of physics themselves have certain symmetries, which lead to conservation laws.

Examples are conservation laws for energy, momentum and angular momentum. We understand now that these symmetries are properties of space-time.

1.3 At a deeper level, symmetries determine the laws of physics.

Symmetries like general co-ordinate invariance and gauge invariance that not only help us find solutions to physical problems, but determine the dynamical laws of physics themselves such as the Einstein, Dirac and Yang-Mills equations. These are the most important cases, but their mathematical study lies beyond this course. I will glimpses of these deeper ideas when possible.

1.4 The basic mathematical idea is a group.

We will give a precise axiomatic definition later. For now, a group describes the transformations (symmetries) of some physical or geometrical object. For example an equilateral triangle ABC can be rotated around its center by 120 degrees to get to an equivalent situation, changing only the labelling of the vertices to BCA . If this do it again we get CAB . A third iteration gets us back to the same triangle, even labelled the same way. The set of these transformations is a group with three elements: the identity 1 (which does nothing), an element ω which rotates by 120degrees, an element ω^2 which rotates by 240 degrees. The condition

$$\omega^3 = 1$$

expresses the fact that a rotation through 360 degrees is the same as the identity. This group $\{1, \omega, \omega^2\}$ is called Z_3 , the cyclic group of three elements. This is an example of an abelian group, one in which the product of two elements does not depend on the order of multiplication. **Abel** was a Norwegian mathematician who, along with **Galois**, invented the idea of a group.

Exercise 1. What is the symmetry group of rotations of a square? Of a rotations of a regular polygon of n sides? Of rotations of a circle?

1.5 In general $ab \neq ba$ in a group

For example, a rigid body (like a book) can be rotated around the x -axis, y -axis or z -axis. It turns out that such rotations do not commute: a rotation by 90 degrees around the x -axis followed by one around the y -axis is not the same thing as first rotating around the y -axis first and then the x -axis. (Try this with a book).

1.6 A related idea is that of a Lie algebra

They are named for Sophus **Lie**, a Norwegian mathematician. A Lie algebra describes infinitesimal symmetries, like rotations through a tiny angle. . If you look at the difference between two such infinitesimal rotations around x and y axes, it is a small rotation around the z axis!. This is expressed by commutation relations

$$L_x L_y - L_y L_x = L_z$$

etc. Such relations define a Lie algebra. These have turned out to be very important in quantum physics. We will study them in some detail.

1.7 Matrices are a useful way of thinking about groups and algebras

Many physical quantities (mass, energy) are represented by numbers. But we have found that rotations cannot be described by numbers: $ab \neq ba$. But matrices can do the job. For example, the rotations through 90 degrees around the x, y, z axes are represented by the matrices

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Exercise 2. Find the multiplication table of the group generated by these matrices; that is, the set of all matrices that can be obtained by multiplying them repeatedly.

1.8 Galois and Abel discovered groups while solving an age old problem: which polynomial equations can you solve algebraically?

Everyone knows that the quadratic equation

$$ax^2 + bx + c = 0$$

has the two solutions

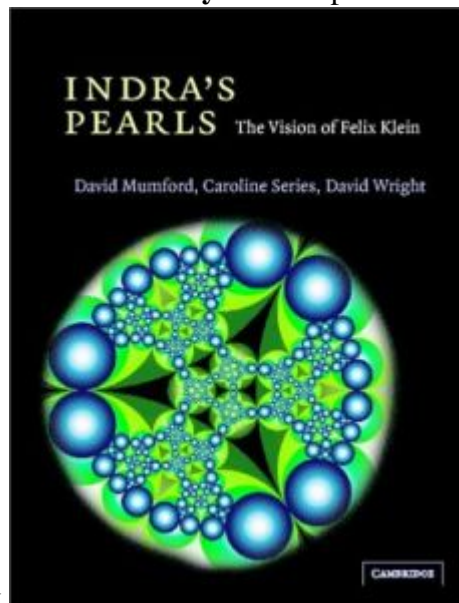
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In the middle ages, it was found that cubic and fourth degree equations can also be solved in a similar way, in terms of cube roots and fourth roots. The formulas are much more complicated though. But no one could find a solution for the general fifth order equation in terms of fifth roots, and similarly for higher order equations. But special cases could be solved that way. Galois, continuing ideas of Abel, showed that in fact this is impossible: there are fifth order equations that cannot be solved even if you could calculate fifth roots. There are similar problems in quantum physics open still: the Schrodinger equations can be solved exactly and others not. How can we tell which are solvable and which cannot be? I have some ideas on this which I will mention later in the course. Can computational complexity be approached this way? $P \neq NP$?

1.9 The Vision of Felix Klein

Before they became standard tools in physics, groups became important in geometry through a visionary research program of **Felix Klein**. It is obvious that a sphere has symmetry under rotations. Also, a lattice of points on the plane separated by equal steps, has a symmetry under translations and some rotations. Klein saw a way to extend these ideas to non-Euclidean geometries such as a hyperboloid. These led Sophus Lie, **Emma Noether**, **Poincare** and others to study deeper connections between groups and physics. It was not until the discovery of relativity and quantum mechanics that the deep role that symmetry plays in physics became

clear. A classic text by **Hermann Weyl** was important in bridging physics and



mathematics. The book

Indra's Pearls; The Vision of Felix Klein by D. Mumford et. al.

gives much more detailed account of the mathematical side of this story. Also many pretty pictures at the book's website

<http://klein.math.okstate.edu/IndrasPearls/>

1.10 Modern physics needs things more general than groups: quantum groups, Hopf algebras, Yangians.

Their mathematical meaning as well as physical consequences are still mysterious. As before they explain why **Bethe** could solve certain really difficult problems (models for magnetism). We would like to systematically solve problems, instead of relying on clever guesses like the Bethe ansatz. This is beyond the scope of this course, but perhaps some of you will join in this quest.

1.11 Cultural references to symmetries can be found long before physics.

Here is an example, from a Buddhist manual on meditation:

In the glistening surface of each pearl are reflected all the other pearls
In each reflection, again are reflected all the infinitely many other pearls,
So that by this process , reflections of reflections continue without end.

Highly symmetric geometric patterns can be found in mosaics of many mosques, on the Taj Mahal as well as in Cathedrals of Europe. We seek symmetry everywhere in life. Even beyond.



