Lecture 10

1. RANDOM MATRICES

The main reference is the book "Random Matrices" by M. L. Mehta. More modern developents are in the book by D. Gioev and P. Deift.

1.1. A matrix whose entries are random variables is a random matrix.

Example 1. This idea was first pursued by Wigner in Nuclear Physics: the thousands of energy levels of nuclei defied any simple dynamical description. The hamiltonian was modelled as a hermitean or real symmetric matrix. Although the actual nuclear eigenvalues are not well described by this model, the level spacing (energy difference between two successive energy levels) fits with those of a random real symmetric matrix. (This happens when the spin-dependent part of the nuclear hamiltonian is important.) Part of the reason for the success is the universality of the spacing: it is the same for a large class of random matrices. This may be viewed as analogous to the central limit theorem, which says that the sum of a large number of random variables is Gaussian for more or less any collection of random variables.

Random matrix theory has turned out to have applications in many other branched of physics and mathematics: zeros of the Riemann zeta funcion of number theory, resistances of wires, statistical inference using Principal Component Analysis and so on. For example let ξ_i for $i = 1, \cdots N$ be a sequence of random variables: prices of stocks in the S&P 500, rain falls in the counties of NY state etc.

Example 2. Imagine we make p measurements of these quantities (e.g., closing stock prices on successive days, rainfalls on successive months). We can arange these into an $n \times p$ rectangular matrix x_{ia}. By subtracting the means values we get another vector:

$$y_{ia} = x_{ia} - \frac{1}{p}\sum_{b=1}^{p} x_{ib}$$

The covariance matrix is

$$\Sigma_{ij} = \frac{1}{p}\sum_{b} y_{ia}y_{ja}$$

This is a positive symmetric matrix of random variables. The eigenvector of the largest eigenvalue of this matrix is important in statistics: it captures most of the random variation (Principal Component Analysis). What is the distribution of this eigenvalue? This answer turns out to be universal (the same for a large class of random variables) and was found by Tracy and Widom.

1.2. **The simplest model of a random matrix is Gaussian.** Here we assume that the matrix elements are independent Gaussians.

1.3. **The Gaussian Unitary Ensemble (GUE) is a random hermitean matrix whose elements have the joint probability distribution function.**

$$\frac{1}{Z}e^{-\mathrm{tr}A^2}dA$$

Here Z is a normalization constant. We assume that the mean of each matrix element is zero. There are N^2 independent real variables in A once the condition of hermiticity is imposed. The p.d.f. is invariant under the action of the unitary group

$$A \to UAU^{\dagger}, \quad A \in U(N)$$

which explains the name.

1.3.1. *The Gaussian Orthogonal Ensemble (GOE) is a random real symmetric matrix with the analogous p.d.f.* This time it is invariant under the orthogonal group action

$$A \to gAg^T, \quad g \in O(N)$$

1.4. **Although the matrix elements are independent variables, the eigenvalues are not.** In particular the eigenvalues exhibit the phenomenon of level repulsion: two eigenvalues of a matrix are unlikely to be close to each other. We can prove this by considering the special case of 2×2 hermitean matrices. Any such matrix can be expanded in terms of the Pauli matrices:

$$A = a_0 1 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 := a_0 + \mathbf{a}\cdot\sigma$$

The eigenvalues are

$$\lambda_{1,2} = a_0 \pm a, \quad a = |\mathbf{a}|$$

The joint pdf of the matrices elements is

$$e^{-\left[a_0^2+|\mathbf{a}|^2\right]}\frac{da_0 d^3\mathbf{a}}{Z}$$

In polar co-ordinates

$$e^{-\left[a_0^2+a^2\right]}a^2\frac{da_0 da}{Z}$$

Thus joint of p.d.f. , ($a = \frac{\lambda_2-\lambda_1}{2}$)

$$e^{-[\lambda_1^2+\lambda_2^2]}|\lambda_1-\lambda_2|^2\frac{d\lambda_1 d\lambda_2}{Z}$$

The extra factor $|\lambda_1-\lambda_2|^2$ is the Jacobian of the change of variables from the matrix elements to the eigenvalues.

1.5. **The joint p.d.f. of the eigenvalues of the GUE is.**

$$P(\lambda_1,\cdots,\lambda_N)=e^{-\sum_i\lambda_i^2}\prod_{i<j}|\lambda_i-\lambda_i|^2\frac{d\lambda_1\cdots d\lambda_N}{Z}$$

The tricky part is to compute tha Jacobian $\prod_{i<j}|\lambda_i-\lambda_i|^2$: the transformation to eigenvalues is analogous to that to polar co-ordinates. This factor can be thought of the volume of the set of all matrices of a given spectrum $\{\lambda_1,\cdots\lambda_N\}$.

$$A=U\text{diag}(\lambda_1,\cdots\lambda_N)U^\dagger$$

$$dA=U\text{diag}(d\lambda_1,\cdots d\lambda_N)U^\dagger+[dUU^\dagger,A]$$

$$\text{tr }dA^2=\sum_i d\lambda_i^2+2\sum_{i,<i}(\lambda_i-\lambda_j)^2|\left[U^\dagger dU\right]_{ij}|^2$$

This is similar to the formula of the metric of Euclidean space in polar co-ordinates. The formula for the p.d.f. above follows.

1.6. **Of special interest is the probability density of a single eigenvalue obtained by integrating all the others out:**

$$R(\lambda_1)=\int e^{-\sum_i\lambda_i^2}\prod_{i<j}|\lambda_i-\lambda_i|^2\frac{d\lambda_2\cdots d\lambda_N}{Z}$$

Remarkably, this approaches a limit as $N\to\infty$. More precisely,

1.7. **The p.d.f. of the normalized variable** $x=\frac{\lambda}{\sqrt{N}}$ **tends to the semi-circular distribution.**

$$\rho(x)=\frac{1}{\pi}\sqrt{2-x^2}$$

In particular, the probability for $|x|>\sqrt{2}$ is zero.

1.8. **Also of interest is the correlation function of a pair of eigenvalues** T_2 **defined by.**

$$R_2(\lambda_1,\lambda_2)=\int e^{-\sum_i\lambda_i^2}\prod_{i<j}|\lambda_i-\lambda_i|^2\frac{d\lambda_{23}\cdots d\lambda_N}{Z}$$

$$T_2(\lambda_2,\lambda_2)=R_2(\lambda_1,\lambda_2)-R(\lambda_1)R(\lambda_2)$$

1.9. **This tends, as $N \to \infty$ to a universal function of the normalized difference $r = \sqrt{2N}|\lambda_2 - \lambda_1|$.**

$$1 - \left[\frac{\sin \pi r}{\pi r}\right]^2$$

Although the formula is originally derived for Gaussians, the correlation turns out to be true for more or less any ensemble of hermitean random matrices.

1.10. **Amazingly, computations show that the zeros of the Riemann zeta function $\zeta(\frac{1}{2} + i\lambda)$ have the same correlation function.** Perhaps this is because the zerosλ_i are the eigenvalues of some hermitean matrix. This gives some strategies for proving the most famous problem in mathematics, the Riemann hypothesis.

1.11. **Tracy and Widom also obtained a universal p.d.f. for the largest eigenvale of a hermitean matrix in terms of the Painleve transcendent of type II.** The Painleve transcendents are a class of six functions that satisfy certain "integrable" ordinary differental equations. Mathematical physicists love them as they interpolate between Airy functions and elliptic functions and have many other beautiful properties.