Lecture 7

1. Addition of Angular Momentum

1.1. **If we add two vectors of lengths r and r' the sum can have any length between $r+r'$ and $|r-r'|$..** In particular if we combine two classical systems with angular momenta j and j' the combined system can have any angular momentun between $j+j'$ and $|j-j'|$. But in quantum mechanics, angular momentum can only take values that are multiples of $\frac{\hbar}{2}$.

1.2. **If the combine two systems with angular momenta j and j' the combined system has angular momenta $j+j', j+j'-1, \cdots |j-j'|$.** To understand this, we need to use the concepts of direct product and direct sum of matrices.

1.3. **The simplest case is adding angular momenta of two spin $\frac{1}{2}$ systems.** The allowed values are $1,0$. The first corresponds to the symmetric states and the second to anti-symmetric states. In more detail, a basis of states for the combined system is

$$|\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle$$

The basis for symmetric states are

$$|\uparrow\uparrow\rangle, \quad \frac{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle}{\sqrt{2}}, \quad |\downarrow\downarrow\rangle$$

These are eigenstates of J_3+J_3' with eigenvalues $1,0,-1$. A more elaborate calculation will show that they are all eigenstates of $(\mathbf{J}+\mathbf{J}')^2$ with eigenvalue $2=1(1+1)$. For example, the state $\uparrow\uparrow\rangle$ is annihilated by both J_+ and J_+' and hence is the highest weight state for $J_+ + J_+'$ as well. Thus

$$(\mathbf{J}+\mathbf{J}')^2|\uparrow\uparrow\rangle = \left[(J_3+J_3')(J_3+J_3'+1)+(J_-+J_-')(J_++J_+')\right]|\uparrow\uparrow\rangle = 2|\uparrow\uparrow\rangle.$$

Exercise 1. Show similarly that $\frac{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle}{\sqrt{2}}, \quad |\downarrow\downarrow\rangle$ are also eigenstates of $(\mathbf{J}+\mathbf{J}')^2$ with eigenvalue 2.

There is only one anti-symmetric state

$$\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}$$

which is an eigenstate of J_3+J_3' with eigenvalue 0.

Exercise 2. Show that $\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}$ is an eigenstates of $(\mathbf{J}+\mathbf{J}')^2$ with eigenvalue 0.

2. SOME NUCLEAR PHYSICS

2.1. **It was found that the nucleus contains an electrically neutral particle in addition to the proton, which has charge one.**

2.1.1. *The proton has a mass of* $m_p = 938$ *MeV and the neutron has mass* $m_n = 939.5$*Mev,* Too close to be a coincidence.

2.1.2. *During beta decay, a neutron converts itself to a proton and an electron.* Also produced is an anti-neutrino which is often hard to detect.

2.1.3. *The atomic number of a nucleus is the number of protons in it. Its atomic mass number is the number of protons plus the number of neutrons.*

$$A = N + Z.$$

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2.2. **Nuclei with the same number of protons but differing numbers of neutrons will form atoms with almost identical chemical properties.**

2.2.1. *This explains the existence of isotopes: atoms with identical chemistry but different masses.* For example, 99.8% of oxygen in nature is the isotope with atomic mass number 16. But oxygen also has stable isotopes of masses 17 and 18and several unstable ones.

Hydrogen has the simplest nucleus with just a single proton. It has a stable isotope of atomic mass number 2 (deuterium) and an unstable isotope (tritium) with atomic mass 3 and a half life of 12.32 years.

2.2.2. *The abundant isotope of Helium has* $Z = 2, A = 4$. *Its nucleus is the alpha particle. Another stable isotope is* $He3$*which is the product of tritium decay.*

2.3. **The binding energy** B **of a nucleus containing** N **neutrons and** Z **protons is** $B(N,Z) = [M(N,Z) - Nm_n - Zm_p]c^2$ **where** $M(N,Z)$ **is the mass of the nucleus.**

2.3.1. *The binding energy of the deuteron is* 2.2 *MeV.*. The condition for stability of a nucleus against beta decay is that $M(N-1, Z+1) - M(N,Z) < m_n + m_e \approx 938.58$ Mev ≈ 1.01 u . Beta decay increases atomic number by one unit and decreases the number of neutrons by one.

2.3.2. *Masses of atoms are measured conveniently in atomic mass units u.* By definition, the Carbon isotope with 6 protons and 6 neutrons and 6 electrons has a mass of 12 amu. Carbon is chosen because it is abundant in natural samples of interest.

$1u = 1.66 \times 10^{-27}$kg $= 931.5$Mev.Thus $m_p \approx 1.007276\,u \quad m_n \approx 1.008665\,u, \quad m_e \approx 0.000594\,u$

2.4. **The particles inside a nucleus are held together by a strong attraction.** Otherwise the nucleus would disintegrate due to Coulomb repulsion. If there are too few neutrons, the nucleus will fission or split up into smaller nuclei. The strong interaction has large biniding energy (few MeV) but has small range (1 fm or 10^{-15}m) which is about the size of a nucleus.

3. ISOSPIN

3.1. **The neutron and proton are different states of the same particle, the nucleon, with different values of a new quantum number called isospin.** Electromagnetism (charge and magnetic moment) and weak interactons responsible for beta decay are small effects in comparsion to the nuclear force. The mass difference, is only about .2%. If we ignore these, the neutron and proton really do look like different states of the same particle.

3.1.1. *Since there are only two possible values for this new quantum number labeling the neutron and the proton, it is analogous to the spin of an electron.* Isospin means 'like spin' in pidgin greek.

3.2. **The nucleon has spin half and isospin half.**

3.3. **When a neutron and a proton combines into a deuteron, they form an isopin 0 state; because of Fermi statistics, the spin must be 1.** The state of a nucleon at rest is a four component complex vector; two such would involve a 4×4 matrix. This matrix must be anti-symmetric because of the exclusion principle, having thus 6 independent states. These can be grouped into sets of three states each that are of spin 1 and isospin zero or isospin one and spin zero. One set of these have an attractive potential and the other must be repulsive. This would explain why there are no *nn* or *pp* nuclei.

3.4. **The α particle is a spin zero and isospin zero state; it can be thought of as a bound state of four nucleons.** The αparticle is the nucleus of the abundant isotope of helium.

3.4.1. *It has a large binding energy: 28.3 MeV.*. Whenever there are the right number of neutrons and protons to form an isospin zero state, the binding energy is unusually large: these are called the 'magic nuclei' and they are usually the end products of fission and fusion reactions.

3.5. **The electromagnetic interactions do not respect isospin symmetry.** In fact, for nucleons, $Q = I_3 + \frac{1}{2}$ where I_3 is isopin.

3.6. **The weak interactions also do not respect isospin symmetry. Nuclear beta decay treats the neutron and the proton differently.**

3.7. **Thus isospin is a symmetry only of strong interactions, which are responsible for the binding of nucleons into nuclei.** A simple formula of Weiszacker for binding energy is found to be surprisingly accurate:

$$B(N,Z) = a_{vol}A - a_{surface}A^{\frac{2}{3}} - a_{Coul}\frac{Z(Z-1)}{A^{1/3}} - a_{sym}\frac{(N-Z)^2}{A}.$$

for some constants a. Here, $A = N + Z$ is the mass number; i.e., the total number of nucleons.

The first term is proportional to the number of nucleons; the second to the surface area, as the density of nuclear matter is roughly constant. The third is the Coulomb repulsion and depends on the number of pairs of protons as well as the inverse of the average distance between them.

The last term is zero if you have equal numbers of neutrons and protons so that we can form an isospin zero combination. It can be explained by the postulate that the nuclear force is independent of isospin and spin states of the nucleons. This is related to the $SU(4)$ model of Wigner.

3.7.1. *Iron with $Z = 26, N = 30$ is one of the most tightly bound nuclei.*

4. The Pi Meson

4.1. **Yukawa suggested that the attractive force among nuclei is due to exchange of a massive particle, of mass $\mu \sim \frac{\hbar}{ac} \sim 100$ MeV.**

4.1.1. *It is useful for conversions to note that $\hbar = 197.3269631(49) MeV\, fm.$.* For simplicity we will for now ignore the fact that there are two kinds of particles (n and p) inside the nucleus. In the next section we will return to this doubling.

4.1.2. *The Klein-Gordon equation with a point source has an exponential decreasing static solution $\phi = g\frac{e^{-\mu r}}{4\pi r}$.* Here g is a constant (Yukawa coupling constant) that measures the strength of the field, analogous to electric charge for the Coulomb field.

4.1.3. *Similar to the photon which mediates the electromagnetic interactions, except the photon is massless and the Coulomb force has infinite range.*

4.1.4. *The exchange of photons can lead to repulsive as well as attractive interactions. Because the nuclear force is always attractive, the spin of the particle must be even. Yukawa suggested it must be spin zero.* Gravity is also always attractive: it is mediated by a hypothetical spin two particle.

4.2. **This particle has since been discovered and is called the π meson.** It has a mass of about 140 MeV. There was some confusion about its discovery. In fact another particle with a very close mass was discovered first in cosmic rays, called the muon. But the muon did not get absorbed by nuclei. It was Marshak (former Chair of our Department) who resolved the confusion: the muon is a lepton, a copy of the electron only with a higher mass. It has no strong interactions with the nuclei. But pions which are caused by cosmic ray collisions in the upper atmosphere decay into the muons, which are detected at lower altitudes.

4.3. **The pi meson has isospin one.** Thus there are three possible isospin states: there are actually three pi mesons, with almost equal masses and electric charges $\pm 1, 0$.

$$\phi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$$

For them the formula for electric charge is

$$Q = I_3.$$

There is no shift, unlike for the fermions. In reality the mass of the charged pions are a few percent different from that of the neutral pion but we ignore that for now. The strong interactions are caused by exchanges of pions:

$$n \to p + \pi^-, \quad p \to n + \pi^+.$$

Because there may not be enough energy to create a free pion in a nucleus, the pions are often virtual: they exist only for a time of order $\frac{1}{\mu}$. But that is enough to produce the attractive interactions of range $\frac{1}{\mu}$.

5. Hadrons

In the 1950s and 1960s experimentalists discovered a whole zoo of strongly interacting particles. They are collectively known as hadrons. Those of half integer spin are called baryons; those of integer spin are called mesons.

5.1. **There is a set of four baryons $\Delta = \begin{pmatrix} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix}$ of spin and isospin both equal to $\frac{3}{2}$..** These decay into a nucleon and a pion. Since the nucleon has $I=\frac{1}{2}, J=\frac{1}{2}$ and the pion $I=1, J=0$ this strong decay respects both spin and isospin conservation.

5.2. **There is a set of three spin one mesons** $\rho = \begin{pmatrix} \rho^+ \\ \rho^0 \\ \rho^- \end{pmatrix}$ **of isospin 1.** Their mass is about two thirds of the mass of a nucleon. They decay strongly into pions.

5.3. **The charges are related to isospin by the relation.**

$$Q = I_3 + \frac{B}{2}.$$

The baryon number is equal to 1 for the half integer spin hadrons (baryons) and equal to zero for mesons.

5.4. **There are hadrons of spins** $J = 0, \frac{1}{2}, 1, \frac{3}{2}2, \cdots$**.** As the spin grows the masses grow approximately proportionately. The high mass hadrons are more and more unstable to decay to lower mass ones. Such very unstable particles are called resonances. As the numbers of hadrons grew into the hundreds, physicists accepted that there must be in principle an infinite number of them.

5.5. **String theory arose as an explanation for the infinitely rising spectrum of hadrons.** A string is a surface in space time whose action is proportonal to its area. Nambu and Goto showed that this implies that the masses of its excited states are proportional to the angular momentum. The Nambu-Goto model only allowed integer spins. Supersymetry was invented by Ramond to include fermions. This may still be a correct explanation, but no one has found a string theory that works in four space time dimensions. The 10 dimensional version of superstring theory is logically consistent and is a candidate for a quantum theory of gravity.

6. Quarks

6.1. **All of the hadrons are bound states of more elementary particles known as quarks.** This is a much simpler explanation for the prolification of hadrons.

6.1.1. *Mesons are bound states of quarks and anti-quarks.*

6.1.2. *Baryons contain three quarks.*

6.1.3. *The baryon number of a quark is* $\frac{1}{3}$..

6.2. **There are a pair of quarks** $\begin{pmatrix} u \\ d \end{pmatrix}$ **forming an isospin** $\frac{1}{2}$ **system.**

6.2.1. *Their charges are given by* $Q = I_3 + \frac{B}{2}$.

$$Q_u = \frac{2}{3}, \quad Q_d = -\frac{1}{3}.$$

6.3. **Quarks have spin $\frac{1}{2}$.** Some group theory will allow us to get the spins and isospins of the hadrons out of those of the quarks.But there is a surprise:

7. THE STATIC QUARK MODEL

7.1. **A first approximation is to treat the hadrons as non-relativistic bound states of quarks.** So the different spin states have the same energy (spin-orbit coupling is a relativistic correction). Combined with isospin this gives an $SU(4)$ symmetry.

7.2. **Quarks are fermions.** It should be impossible to put three up quarks into a state of spin $\frac{3}{2}$. But then how do we explain the Δ^{++}? One idea was that quarks obeyed some exotic statistics that violates the Pauli exclusion principle. Another possibility is that there is a extra degree of freedom.

7.3. **Each quark comes in three colors.** Thus there are three states for the up quark (not counting the spin states) and three for the down quark.The word color is used in a figurative way here: this quantum number has nothing at all to do with light: nothing to do with electromagnetism.

7.4. **There is an SU(3) symmetry corresponding to rotations among the color states.** Since quarks of different colors have the same masses, isospin, charges etc.

7.5. **Hadrons are color neutral.** Nucleons and mesons do not have this extra degree of freedom: we would have seen this in nuclear physics. Hadron states are invariant under the color SU(3) symmetry. This means that color cannot be directly measured:it can be inferred indirectly from properties of hadrons.

7.6. **The grond state of a three quark system must be a symmetric combination of three fundamental representations of $SU(4)$.** There are $\frac{4(4+1)(4+2)}{3!} = 20$ such states. These can be split into $I = \frac{3}{2}, J = \frac{3}{2}$ and $I = \frac{1}{2}, J = \frac{1}{2}$ states. The first are the Δ and the second set the nucleons. There are $4 \times 4 = 16$ states for the Δ and $2 \times 2 = 4$ states for the nucleon which add up to twenty.

7.7. **The protons and neutron have anamolous magnetic moments.** The magnetic moment points along the angular momentum. For a fundamental spin half particle their ratio is predicted by the Dirac equation to be twice the Bohr magneton $\frac{e\hbar}{2m}$. Instead the proton has magnetic moment approximately equal to 3 and the neutron has -2 (The negative sign means that it points opposite to the spin. The charge unit is taken to be equal to the charge of the proton.) This strange fact can be explained by the quark model and was one of its early successes. We can also predict magnetic moments for the Δ but they are harder to measure as the particles are unstable.

7.8. **The quarks have magnetic moments of fundamental particles.** Dirac's theoy predicts them to be

$$\mu_u = 2\frac{e_u\hbar}{m_u}, \quad \mu_d = 2\frac{e_d\hbar}{m_d}$$

In the static quark model

$$m_u = m_d = \frac{m}{3}$$

where m is the mass of the nucleon. Also charge is given by the isospin with $B = \frac{1}{3}$Thus the magnetic moment operator of a quark is

$$\mu = \frac{2e}{\frac{m}{3}}\left[\frac{\tau_3}{2} + \frac{1}{6}\right]\frac{\sigma}{2} = \frac{3e}{2m}\left[\tau_3 + \frac{1}{3}\right]\sigma.$$

The total magnetic moment of a baryon is given by summing over the three quarks

$$\frac{3e}{2m}\sum_{a=1}^{3}\left[\tau_{3a} + \frac{1}{3}\right]\sigma_a$$

The second term can be written in terms of the total spin

$$\frac{e}{m}\mathbf{S}.$$

The first term is (considering just the third component of the magnetic moment)

$$\frac{6}{m}\sum_a \frac{\tau_{3a}}{2}\frac{\sigma_{3a}}{2}$$

It is now possible to get nucleon magnetic moments out of this.See *Introduction to High Energy Physics* by Perkins for details.

7.9. **Isospin breaking in the baryon masses can be explained by unequal masses for the up and down quarks.** The nucleon and Delta masses can be thought of as made of an average value that is independent of I_3 plus a piece proprtional to it

$$M = M_0 + \varepsilon I_3$$

This means that the differences between masses of states with different I_3 equal to ± 1 are constant. Such simple formulas gave early indications of the validity of the quark model. More elaborate potential models were developed but did not improve matters for the up and down quarks. They work very well for the charm and b quarks.

Exercise 3. Derive a relation between the mass differences of the Δparticles within the static quark model. Compare with experimental data.