GRAVITATION F10

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Lecture 10

1. Curvature

1.1. **If the metric $g_{\mu\nu}$ is a constant, the Christoffel symbols vanish and the geodesics are straightlines.** Thus the geometry is locally that of Euclidean space.

1.1.1. *But just because $g_{\mu\nu}$ depends on co-ordinates it does not follow that the space is curved: we could be using a curvilinear co-ordinate system.* How can we tell if a change of co-ordinates can bring the metric to a constant? If there is a tensor that vanishes in flat space but not in curved space, we would have such a criterion. The Christoffel symbols do not transform as a tensor so they won't do the job. The correct quantity was discovered by Riemann.

1.1.2. *Recall that the commutator of partial derivatives is zero.*

$$\partial_\mu \partial_\nu v^\rho - \partial_\nu \partial_\mu v^\rho = 0$$

1.1.3. *The commutator of covariant derivatives of a vector field is not always zero.*

1.2. **The curvature tensor is defined in terms of the commutator of covariant derivatives.**

$$D_\mu D_\nu v^\rho - D_\nu D_\mu v^\rho = R^\rho_{\mu\nu\sigma} v^\sigma$$

The important observation is that the commutator does not involve derivatives of v: they all cancel out.

1.2.1. *By direct calculation.*

$$R^\rho_{\mu\nu\sigma} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\sigma}$$

We already know Γ in terms of derivatives of $g_{\mu\nu}$. Thus $R^\rho_{\mu\nu\sigma}$ is completely determined by the metric tensor and its derivatives up to second order.

Remark 1. There are many different conventions on where to place the indices of the curvature tensor. Especially if you are using formulas from different sources, you should check carefully the definitions of the curvature tensor they use. Our convention here is the same as in Landau-Lifshitz, but to see that they agree you have to use the symmetry properties of the curvatute tensor given below.

1.2.2. *We will also find it convenient to define a version of the curvature tensor with all covariant indices.*

$$R_{\mu\nu\rho\sigma} = g_{\rho\alpha} R^{\alpha}_{\mu\nu\rho}$$

By explicit, but tedious, calculation

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left[\partial_\nu\partial_\rho g_{\mu\sigma} - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\mu\partial_\sigma g_{\nu\rho}\right] + g_{\alpha\beta}\Gamma^{\alpha}_{\nu\rho}\Gamma^{\beta}_{\mu\sigma} - g_{\alpha\beta}\Gamma^{\alpha}_{\mu\rho}\Gamma^{\beta}_{\nu\sigma}$$

1.3. **If there is a co-ordinate system in which the metric tensor is a constant, the curvature tensor will vanish in any co-ordinate system.** If the metric tensor is a constant in some system, the curvature vanishes in that system. But $R^{\mu}_{\nu\rho\sigma}$ is a tensor: if it is vanishes in one system it vanishes in any system.If you transform to a different system in which the metric is no longer a constant, the various terms involving derivatives above can be non-zero. But they will all cancel out.

The converse is true as well. To see this, we first define

1.4. **A vector field is covariantly constant if.**

$$D_\mu v^\nu = 0$$

Such vector fields may not exist. By taking second derivatives and taking the anti-symmetric part we get an integrabilty condition:

$$R^{\nu}_{\mu\rho\sigma} v^\sigma = 0$$

Thus a covariantly constant vector fields is a sort of zero eigenvector of the curvature tensor: there are metrics for which the curvature tensor does not vanish in direction, so such a vector field may not exist. Conversely, if the curvature tensor is zero, there are as many linearly independently solutions as the dimension of the space. In this situation, we can form a co-ordinate system where each axis is tangential to a covariantly constant vector field. But in such a system, the metric tensor (inner product of vectors along the axes) will be a constant.

1.5. **If the curvature tensor vanishes, there is a co-ordinate system in which the metric tensor is a constant.** Thus every point has a neighborhood in which the space would look Euclidean.

2. Properties of the Curvature Tensor

2.1. **The curvature tensor has many symmetries under the interchange of indices.** The first is obvious from the definition

$$R^{\sigma}_{\mu\nu\rho} = -R^{\sigma}_{\nu\mu\rho}$$

The explicit formula in terms of the second derivatives of the metric for covariant form of the curvature gives

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$$

and also

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

Milnor found a convenient way to summarize them.

2.1.1. *The curvature tensor defines a biquadratic form.*

$$R(u,v) = R_{\mu\nu\rho\sigma} u^\mu v^\nu u^\rho v^\sigma$$

A biquadratic is a function of two vectors satisfying

$$Q(u,v) = Q(v,u), \quad Q(\lambda u, v) = \lambda^2 Q(u,v), \quad Q(u,u) = 0$$

The curvature form satsifies these conditions.

2.2. **In addition it satisfies the differential identities of Bianchi.**

$$D_\mu R^\alpha_{\nu\rho\sigma} + D_\nu R^\alpha_{\rho\mu\sigma} + D_\rho R^\alpha_{\mu\nu\sigma} = 0$$

2.2.1. *These identities are similar to those satisfied by the Maxwell tensor.*

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

which follow from its definition

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Ironically, the much more complicated case (e.g., non-linear) of the curvature tensor was discovered first.

3. Geodesic Deviation

An infinitesimal change of a curve (a deviation) is given by a vectir field along it: connect the two points on each of the curves corresponding to the same parameter value. Suppose we change a geodesic slightly so that the new curve is also a geodesic; i.e., to infinitesimally close geodesics. The vector field describing the deviation satisfies a differential equation involving curvature. It helps us to understand the geometric meaning of curvature.

3.1. **Infinitesimal deviations of geodesics are measured by curvature.**

$$\frac{D^2 v^\mu}{d\tau^2} + R^\mu_{\nu\rho\sigma} v^\nu \dot{x}^\rho \dot{x}^\sigma = 0$$

Here D^2 denotes the second co-variant derivative along the geodesic:

$$\frac{Dv^\mu}{d\tau} \equiv \frac{dv^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho v^\sigma$$

$$\frac{D^2 v^\mu}{d\tau^2} \equiv \frac{d}{d\tau}\left[\frac{dv^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho v^\sigma\right] + \Gamma^\mu_{\alpha\nu}\dot{x}^\alpha \left[\frac{dv^\nu}{d\tau} + \Gamma^\nu_{\rho\sigma}\dot{x}^\rho v^\sigma\right]$$

v^μ is a vector field that at each point describes how the geodesic is changed. In the hands of Jacobi this equation became a powerful tool to extract geometric information about a metric. If the metric is a constant, the curvature is zero and geodesics that start out parallel remain parallel: the distance between points with the same parameter remains constant.

3.2. **If the curvature is zero, nearby geodesics remain parallel.** Two nearby freely falling observers will continue to move at constant velocity relative to each other as long as the curvature is zero.