GRAVITATION F10

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Lecture 11

1. The Wave Equation

1.1. **The amplitude of a small wave propagating with speed c satisfies.**

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0$$

1.1.1. *Plane waves are solutions*

$$\phi(x) = e^{i[\omega t - \mathbf{k}\cdot\mathbf{x}]}, \quad \frac{\omega^2}{c^2} - \mathbf{k}^2 = 0$$

.

1.2. **In Lorentz invariant form the wave equation is.**

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

Remember that all wave equations are invariant under Lorentz transformations; even sound. But there is something special about light: the speed is the same for all observers. Relativity is much more than invariance under Lorentz transformations.

1.3. **The wave equation follows from a variational principle.**

$$S = \frac{1}{2}\int \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi dx$$

1.3.1. *Here dx stands for the volume measure of space-time $dx^0dx^1dx^2dx^3$.* Just like in mechanics, except that the function depends on several variables.

$$\delta S = \int \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi dx = \int \partial_\nu\left[\eta^{\mu\nu}\partial_\mu\phi\delta\phi\right]dx - \int\left[\eta^{\mu\nu}\partial_\nu\partial_\mu\phi\right]\delta\phi dx$$

By using Gauss' theorem (that the integral of the divergence of a vector field is equal to flux through the boundary) the first term depends only on the boundary. We assume that the variation $\delta\phi = 0$ at the boundary; this is analogous to requiring that the variation should vanish at the initial and final points in mechanics. Thus the condition that $\delta S = 0$ is the wave equation.

1.4. **Under nonlinear change of co-ordinates the volume measure changes by the Jacobian determinant.** Recall that the Jacobi matrix appears in the infinitesimal change of co-ordinates

$$dx'^{\nu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}} dx^{\mu} \equiv J^{\nu}_{\mu} dx^{\mu}$$

and that the change in volume measure involves the Jacobian

$$dx' \equiv dx^0 dx^1 dx^2 dx^3 = \det J dx$$

1.5. **The determinant of the metric tensor transforms with the square of the Jacobian.**

$$g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}, \quad g' = J^{-1} g J^{-1T}$$

$$\det[g'] = [\det J]^{-2} \det g$$

1.5.1. *The metric tensor of space-time has negative determinant.* There are three negative eigenvalues (space) and one positive eigenvalue (time).

1.6. **The combination $\sqrt{-\det g} dx$ is invariant under co-ordinate transformations.** The determinants cancel out. If the metric is positive we would not put in the negative sign.

1.6.1. *In spherical polar co-ordinates.*

$$ds^2 = dr^2 + r^2 \left[d\theta^2 + \sin^2\theta d\phi^2 \right]$$

$$\sqrt{g} dx = r^2 \sin\theta dr d\theta d\phi$$

1.7. **The generally covariant version of the action for the wave equation is.**

$$S = \frac{1}{2} \int g^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi \ \sqrt{-\det g} \ dx$$

The combination $g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi$ is a scalar: is invariant under co-ordinate changes. The last part $\sqrt{-\det g} \ dx$ is invariant as well.

1.8. **The generally covariant version of the wave equation is.**

$$\partial_{\mu} \left[\sqrt{-\det g} g^{\mu\nu} \partial_{\nu}\phi \right] = 0$$

As above

$$\delta S = \int g^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\delta\phi \ \sqrt{-\det g} \ dx = \int \partial_{\nu} \left[g^{\mu\nu} \partial_{\mu}\phi \delta\phi \ \sqrt{-\det g} \right] dx - \int \partial_{\nu} \left[g^{\mu\nu} \partial_{\mu}\phi \ \sqrt{-\det g} \right] \delta\phi dx$$

Again the first term is zero because $\delta\phi = 0$ on the boundary.

1.8.1. *But we could have obtained a generally covariant wave equation by replacing partial derivatives by covariant derivatives.*

$$g^{\mu\nu} D_{\mu} D_{\nu}\phi = 0$$

1.8.2. *This happens to be equivalent to the one above.*

$$\frac{1}{\sqrt{-\det g}}\partial_\mu\left[\sqrt{-\det g}g^{\mu\nu}\partial_\nu\phi\right] = g^{\mu\nu}D_\mu D_\nu\phi$$

Proof. First, recall that the covariant derivative and partial derivative are the same for a scalar. Thus

$$g^{\mu\nu}D_\mu D_\nu\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi - g^{\mu\nu}\Gamma^\rho_{\mu\nu}\partial_\rho\phi$$

Now,

$$g^{\mu\nu}\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\left[\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}\right]$$

$$= g^{\mu\nu}\partial_\mu g_{\sigma\nu}g^{\rho\sigma} - \frac{1}{2}g^{\rho\sigma}\left[g^{\mu\nu}\partial_\sigma g_{\mu\nu}\right]$$

Next, recall that the infinitesimal variation of the inverse of a matrix is related to its own variation by

$$d[A^{-1}] = -A^{-1}[dA]A^{-1}$$

Thus

$$g^{\mu\nu}\partial_\alpha g_{\sigma\nu}g^{\rho\sigma} = -\partial_\alpha g^{\mu\rho}$$

and

$$g^{\mu\nu}\partial_\mu g_{\sigma\nu}g^{\rho\sigma} = -\partial_\mu g^{\mu\rho}$$

On the other hand the variation of the determinant of a matrix can be calculated using

$$\log\det A = \operatorname{tr}\log A$$

$$\frac{\partial_\mu \det A}{\det A} = \operatorname{tr}A^{-1}\partial_\mu A$$

Thus

$$g^{\mu\nu}\partial_\sigma g_{\mu\nu} = \partial_\mu \log[-\det g]$$

(Switching the sign only shifts the log by a constant.) and

$$\frac{1}{2}\left[g^{\mu\nu}\partial_\sigma g_{\mu\nu}\right] = \partial_\mu \log\sqrt{-\det g} = \frac{\partial_\mu\sqrt{-\det g}}{\sqrt{-\det g}}.$$

Pulling all this together

$$g^{\mu\nu}D_\mu D_\nu\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi + \left[\partial_\mu g^{\mu\rho}\right]\partial_\rho\phi + \frac{1}{\sqrt{-\det g}}\left[\partial_\mu\sqrt{-\det g}\right]g^{\mu\rho}\partial_\rho\phi.$$

The r.h.s. is the same as

$$\frac{1}{\sqrt{-\det g}}\partial_\mu\left[\sqrt{-\det g}g^{\mu\rho}\partial_\rho\phi\right]$$

expanded out. □

1.9. **The wave equation in curved space time is.**

$$g^{\mu\nu}D_\mu D_\nu \phi = 0$$

For calculations the equivalent form $\frac{1}{\sqrt{-\det g}}\partial_\mu\left[\sqrt{-\det g}g^{\mu\rho}\partial_\rho\phi\right]$=0 is more convenient.

2. Maxwell's Equations in Curved Space-Time

2.1. **Recall that Maxwell equations in Lorentz covariant form are.**

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

2.2. **They follow from the variational principle.**

$$S = \frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}dx + \int j^\mu A_\mu dx$$

First,

$$\delta S = \int F^{\mu\nu}\partial_\mu \delta A_\nu dx + \int j^\nu \delta A_\nu dx$$

Now integrate by parts the first term.

2.3. **This leads to a wave equation with source for the electromagnetic potential.**

$$\partial_\mu\partial^\mu A^\nu - \partial^\nu\left[\partial_\mu A^\mu\right] = j^\nu$$

It is common to impose the condition $\partial_\mu A^\mu = 0$,(the Lorentz gauge) taking advantage of the gauge invariance $A_\mu \mapsto A_\mu + \partial_\mu \Lambda$. Then each component of A_μ satisfies the wave equation

$$\partial_\mu\partial^\mu A^\nu = j^\nu$$

2.4. **The generally covariant form of Maxwel's equations is.**

$$D_\mu F^{\mu\nu} = j^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Recall that the Christoffel symbols cancel out in the antisymmetric derivative of a covariant vector.

2.5. **In terms of potentials.**

$$D_\mu D^\mu A^\nu - D_\mu D^\nu A_\mu = j^\mu$$

We cannot interchange the derivatives in the second term without introducing some terms involving curvature.

2.6. **An equivalent form of the curved space Maxwell's equations is.**

$$\frac{1}{\sqrt{-\det g}}\partial_\mu\left[\sqrt{-\det g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}\right] = j^\nu$$

2.7. **This follows from the covariant variational principle.**

$$S = \frac{1}{4}\int F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\sqrt{-\det g}dx + \int j^\mu A_\mu\sqrt{-\det g}dx$$

2.8. **These equations tell us how the gravitational field affects the propagation of light.** For example it can tell us how light is diffracted and refracted by a gravitational field. Spectacular phenomena such as gravitational lensing follow from this. More later.