GRAVITATION F10

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Lecture 12

1. Maxwell's Equations in Curved Space-Time

1.1. **Recall that Maxwell equations in Lorentz covariant form are.**

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

1.2. **They follow from the variational principle.**

$$S = \frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}dx + \int j^\mu A_\mu dx$$

First,

$$\delta S = \int F^{\mu\nu}\partial_\mu \delta A_\nu dx + \int j^\nu \delta A_\nu dx$$

Now integrate by parts the first term.

1.3. **This leads to a wave equation with source for the electromagnetic potential.**

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu [\partial_\mu A^\mu] = j^\nu$$

It is common to impose the condition $\partial_\mu A^\mu = 0$,(the Lorentz gauge) taking advantage of the gauge invariance $A_\mu \mapsto A_\mu + \partial_\mu \Lambda$. Then each component of A_μ satisfies the wave equation

$$\partial_\mu \partial^\mu A^\nu = j^\nu$$

1.4. **The generally covariant form of Maxwel's equations is.**

$$D_\mu F^{\mu\nu} = j^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Recall that the Christoffel symbols cancel out in the antisymmetric derivative of a covariant vector.

1.5. **In terms of potentials.**

$$D_\mu D^\mu A^\nu - D_\mu D^\nu A_\mu = j^\mu$$

We cannot interchange the derivatives in the second term without introducing some terms involving curvature.

1.6. **An equivalent form of the curved space Maxwell's equations is.**

$$\frac{1}{\sqrt{-\det g}}\partial_\mu \left[\sqrt{-\det g}\, g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}\right] = j^\nu$$

1.7. **This follows from the covariant variational principle.**

$$S = \frac{1}{4}\int F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\sqrt{-\det g}dx + \int j^{\mu}A_{\mu}\sqrt{-\det g}dx$$

1.8. **These equations tell us how the gravitational field affects the propagation of light.** For example it can tell us how light is diffracted and refracted by a gravitational field. Spectacular phenomena such as gravitational lensing follow from this. More later.

2. Conservation Laws

2.1. **The electric current is a Lorentz vector field j_μ satisfying.**

$$D_\nu j^\nu = 0$$

2.1.1. *This follows from Maxwel's equations.*

$$D_\mu F^{\mu\nu} = j^\nu$$

First,

$$D_\nu D_\mu F^{\mu\nu} = D_\nu j^\nu$$

By antisymmetry, l.h.s.

$$[D_\nu, D_\mu]F^{\mu\nu} = R^{\mu}_{\nu\mu\rho}F^{\rho\nu} + R^{\nu}_{\nu\mu\rho}F^{\mu\rho} = R_{\nu\rho}F^{\rho\nu} + R_{\mu\rho}F^{\mu\rho}$$

The contraction

$$R_{\nu\rho} = R^{\mu}_{\nu\mu\rho}$$

is called the Ricci tensor. It plays an important role in GR. For now, we just need that it is symmetric (consequence of the symmetries of the Riemann tensor).

2.2. **An equivalent form of the divergence of a vector field is.**

$$D_\mu j^\mu = \partial_\mu\left[\sqrt{-\det g}j^\mu\right]$$

This follows as for the wave equation

2.3. **The conservation of electric charge follows.** Consider a region in space-time bounded by two space-like surfaces $x^0 = T_1, x^0 = T_2$, with $T_2 > T_1$. The integral of $D_\mu j^\mu$ in this region can (using Gauss's theorem)

$$\int_{T_1} j^0\sqrt{-\det g}dx^1dx^2dx^3 = \int_{T_2} j^0\sqrt{-\det g}dx^1dx^2dx^3$$

(We are assuming that the electric current vanishes at spatial infinity.) Thus j^0 integrated with respect to the invariant volume measure is a conserved quantity. Such conservation laws are very important and we should understand them in several different ways.

2.4. **Charge conservation follows from gauge invariance.** Under the gauge transformation

$$A_\mu \mapsto A_\mu + \partial_\mu \Lambda$$

the change in the Maxwell action is

$$\int j^\mu \sqrt{-\det g}\partial_\mu \Lambda dx = -\int \left\{\partial_\mu \left[\sqrt{-\det g} j^\mu\right]\right\} \Lambda dx$$

When the equation of motion is satisfied the action should be unchanged under all infinitesimal transformations. Gauge transformations produce a change that does not involve the electromagnetic field, only its source. Hence the source must satisfy the identity

$$\partial_\mu \left[\sqrt{-\det g} j^\mu\right] = 0$$

if the action is to be extremal.

2.4.1. *The electric current can be defined as the variation of the 'source' action w.r.t. the potential.*

$$j^\mu \sqrt{-\det g} = \frac{\delta S_1}{\delta A_\mu}, \quad S_1 = \int A_\mu j^\mu \sqrt{-\det g} dx$$

2.5. **The electric charge is the integral of electric current over a space-like surface.** A surface is space-like if its normal vector is time-like. The integral

$$Q = \int j^\nu \sqrt{-\det g} dS_\nu$$

is a scalar quantity equal to the total electric charge in that region of space. The integral of the electric current on a surface whose normal is space-like has another meaning: it is the flux of electric charge through that surface.

3. The Stress Tensor

3.1. **There is a tensor field $T^{\mu\nu}$ whose integral over a space-like surface is energy-momentum.**

$$P^\mu = \int T^{\mu\nu} \sqrt{-\det g} dS_\nu$$

Thus T^{00} is energy density and T^{i0} is momentum density. Of course, energy density includes mass as well as all other forms of energy. What is the meaning of the remaining components?

3.2. **The integral over a surface with space-like normal is the total flux of energy or momentum across that surface.** If an electromagnetic wave is propagating through a region, it carries some energy and momentum across such a surface. The component T^{ij} is the amount of the ith component of momentum carried across a small surface whose normal is pointed in the jth direction. This can also be thought of the force felt on that surface per unit area (stress) . Also T^{0j} is the energy carried across this surface, or the work being done by moving that surface an infinitesimal amount in the jth direction.

Thus the tensor $T^{\mu\nu}$ combines the mechanical notions of energy density, momentum density, stress into a single entity.

3.3. **The stress tensor is the variation of matter action w.r.t. the metric tensor.** The energy and momentum denisty of matter, including the e.m. field is the source of gravity. So the variation w.r.t. $g_{\mu\nu}$ will give the stress tensor.

$$T^{\mu\nu}\sqrt{-\det g} = \frac{\delta S_m}{\delta g_{\mu\nu}}$$

It follows that

3.4. **The tensor field $T^{\mu\nu}$ is symmetric.**

3.5. **The stress tensor of a massless scalar field follows from its action.**

$$S_\phi = \frac{1}{2}\int g^{\mu\nu}\sqrt{-\det g}\partial_\mu\phi\partial_\nu\phi dx$$

Recall that

$$\delta \log\det g = \delta \mathrm{tr}\log g = g^{\mu\nu}\delta g_{\mu\nu}$$

It is actually a bit more convenient to vary w.r.t. the inverse matrix $g^{\mu\nu}$

$$\delta\log\det g = -g_{\mu\nu}\delta g^{\mu\nu}, \implies \delta\sqrt{-\det g} = -\frac{1}{2}\sqrt{-\det g}g_{\mu\nu}\delta g^{\mu\nu}$$

Thus

$$\delta S_\phi = \frac{1}{2}\int \delta g^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\right]\sqrt{-\det g}dx$$

and

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi$$

3.5.1. *The energy density is positive in Minkowski space.*

$$T_{00} = \frac{1}{2}\left\{[\partial_0\phi]^2 + [\partial_i\phi]^2\right\}$$

3.6. **The field equations imply that the stress tensor is conserved.**

$$D_\mu T^{\mu\nu} = 0$$

$$D^\mu\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\right] = \partial_\mu\phi D^\mu\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}D^\mu\left[\partial_\rho\phi\partial_\sigma\phi\right]$$

But

$$g^{\rho\sigma}D_\nu\left[\partial_\rho\phi\partial_\sigma\phi\right] = 2g^{\rho\sigma}\left[D_\rho D_\nu\phi\right]\partial_\sigma\phi = 2\left[D^\mu\partial_\nu\phi\right]\partial_\mu\phi$$

so that the two terms cancel.

3.7. **The stress-tensor of the electromagnetic field also can be computed from its action.** Ignoring sources for now,

$$S = \frac{1}{4}\int F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\sqrt{-\det g}dx$$

$$\delta S = \frac{1}{2}\int F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}\delta g^{\nu\sigma}\sqrt{-\det g}dx - \frac{1}{8}\int F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\sqrt{-\det g}g_{\alpha\beta}\delta g^{\alpha\beta}dx$$

so that

$$T_{\mu\nu} = \frac{1}{2}F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} - \frac{1}{8}g_{\mu\nu}g^{\rho\sigma}g^{\beta}F_{\rho\alpha}F_{\sigma\beta}$$

3.7.1. *The energy density T_{00} and momentum density T_{0i} in flat space are familiar special caes.*

$$\frac{1}{2}\int \left[\mathbf{E}^2 + \mathbf{B}^2\right] d^3x, \qquad \int \mathbf{E}\times\mathbf{B}d^3x$$

3.8. **Conservation of the stress tensor follows from Maxwell's equations.**

$$D_\mu T^{\mu\nu} = 0$$

3.8.1. *The stress tensor of the electromagnetic field is also traceless.*

$$g^{\mu\nu}T_{\mu\nu} = 0$$