GRAVITATION F10

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Lecture 13

1. The Vacuum Einstein's Equation

1.1. **The metric tensor of space-time satisfies a nonlinear Partial Differential Equation which determines it given initial conditions.** The metric tensor describes the gravitational field. In the case of the electromagnetic field, Maxwell's equations along with the initial value of **E** and **B** determine the field for all future. There must be an analogous set of equations for the gravitational field as well. They were discovered by Einstein. We will study them first in the case without sources, then introduce sources.

The most important complication we must deal with is that the Einstein equations are non-linear. Physically this is because the source of gravity is mass-energy; but the gravitational field carries potential energy. Thus gravity can be its own source: nonlinearity.

Mathematically also this is clear because there is no meaning to adding metric tensors, unlike say electromagnetic fields. A linear combination of two electric fields can be an elecric field. But a linear combination of two metric tensors may not be a metric tensor: it may not have one positive and three negative eigenvalues. Thus the space of allowed values of the field is already not a linear space in the case of gravity: it does not even make sense to require that the equations be linear.

1.2. **These equations have to be generally covariant; i.e., tensor equations.** The whole basis of General Relativity is its covariant under arbitrary changes of co-ordinates. The equations that determine the metric must hold in any co-ordinate system or reference frame.

There is no tensor that can be constucted from the first derivatives of the metric tensor; e.g., the Christoffel symbols are not tensors. The only tensors made of the second derivatives of the metric are the curvature tensor and its various contractions.

1.3. **The equations must involve second derivatives of the metric.** Maxwell's equations involve second order derivatives of the e.m. potential. It seems reasonable that the gravitational field also satisfy an analogous, but nonlinear, equation. A static solution can then yield Newton's inverse square law just as the Coulomb solution follows from Maxwell's equation.

1.4. **It is too strong a condition to impose that the curvature tensor vanish.** Although this is a econd order PDE, it is not the right choice. If the curvature tensor is zero, there is a co-ordinate system in which the metric tensor is constant, so that geodesics are straightlines. But then there is no gravitational field. The problem is that the curvature tensor has many more independent components than the metric tensor. (Twenty vs ten). So we seek a contraction of the curvature tensor with as many independent components as the metric tensor.

1.5. **The Ricci tensor is defined as the contraction of the Riemann tensor.**

$$R_{\mu\rho} = R^{\nu}_{\mu\nu\rho}$$

It is symmetric because the Riemann tensor satisfies

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

If we look up the formula for the Riemann tensor in terms of the metric tensor we can see that

$$R_{\mu\rho} = \frac{1}{2} g^{\sigma\nu} \left[\partial_\nu \partial_\rho g_{\mu\sigma} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\mu \partial_\sigma g_{\nu\rho}\right] + \left[g^{\sigma\nu} g_{\alpha\beta} \Gamma^{\alpha}_{\nu\rho} \Gamma^{\beta}_{\mu\sigma} - g_{\alpha\beta} \Gamma^{\alpha}_{\mu\rho} \Gamma^{\beta}_{\nu\sigma}\right]$$

Thus upto nonlinear terms it involves some kind of wave operator acting on the metric tensor. To make this clearer let us assume that the deviation from Minkowski metric is small:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| << 1$$

Then

$$g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}, \quad h^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma}$$

and the terms involving Γ^2 in $R_{\mu\nu}$ are second order. Thus to first order

$$R_{\mu\rho} \approx \frac{1}{2} \eta^{\sigma\nu} \left[\partial_\nu \partial_\rho h_{\mu\sigma} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho} + \partial_\mu \partial_\sigma h_{\nu\rho}\right]$$

$$= \frac{1}{2} \left[-\Box h_{\mu\rho} + \partial_\rho \partial_\nu h^{\nu}_{\mu} + \partial_\mu \partial_\nu h^{\nu}_{\rho} - \partial_\mu \partial_\rho h^{\nu}_{\nu}\right]$$

This is similar to the wave operator that arises from Maxwell's equations

$$\partial^\mu F_{\mu\nu} = \Box A_\nu - \partial_\nu \partial_\mu A^\mu$$

If we impose the gauge condition $\partial_\mu A^\mu = 0$ this is the usual wave operator. In the same way of if we could impose $\partial_\nu h^{\nu}_{\mu} = 0$ and $h^{\nu}_{\nu} = 0$ (may be by a choice of co-ordinate system) the Ricci tensor would be just the wave operator acting on the metric petrurbation $h_{\mu\nu}$. Thus we come around to the idea that imposing that the Ricci tensor be zero in the vacuum is a reasonable field equation for gravity.

1.6. **Einstein's equation in the vacuum is the vanishing of the Ricci tensor.**

$$R_{\mu\nu} = 0.$$

We will see later how the presence of sources (such as matter fields or a cosmological constant) modifies this equation.

1.7. **Minkowski space is a solution.** In the absence of all sources, there must be a solution that simply describes flat space, without any gravitational field.

1.8. **There are solutions that are small perturbations from Minkowski space propagating as gravitational waves.**

$$\Box h_{\mu\rho} - \partial_\rho \partial_\nu h^\nu_\mu - \partial_\mu \partial_\nu h^\nu_\rho + \partial_\mu \partial_\rho h^\nu_\nu = 0$$

Here we already see a far reaching prediction of General Relativity: gravity can propagate as a wave just like light. The gravitational waves are described by a tensor and not a vector field. If they are of small amplitude they will move at the velocity of light.