GRAVITATION F10

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Lecture 15

1. Centrally Symmetric Metrics

1.1. **Schwarschild discovered a centrally symmetric and static solution to the vacuum Einstein's equations.** Originally this was thought of as describing the exterior of a star: when the distance to the center is comparable to the size of the star, this soultion would be patched on to another that includes the effects of sources. Later on it was realized that the solution makes sense even without any sources anywhere: it describes a completely new phenomenon, a Blackhole. In a sense this is the analogue of the Coulomb solution of Maxwell's equations. As in the Coulomb solution, there is a singularity at the origin, where the curvature goes to infinity. But unlike in the Coulomb solution, an observer at a large distance cannot see this singularity. It is hidden behind an event horizon, a surface through which nothing, not even light, can escape. Thus in the region visible to an external observer, the solution is smooth. The blackhole is one of the strangest predictions of General Relativity.

1.2. **It is the only centrally symmetric solution of vacuum Einstein's equations.** It turns out that once you assume spherical (central) symmetry, the vacuum solution has to be static. This is one of the early theorems in GR, due to Birkhoff. In particular this means that a spherically symmetric source of finite extent cannot emit gravitational radiation: the solution external to it must be static. There are many variable stars that undergo a radial oscillation. These are not sources of gravitational radiation; indeed their external gravitational field is exactly the same as that of a point mass. In a sense this is a generalization of Newton's result that external gravitational field of a spherically symmetric object is that of a point mass located at the center. But Newton's theory did not allow for gravitational radiation anyway. We will prove Birkhoff's theorem as part of the solution of the centrally symmetric Einstein's equations.

1.3. **Any centrally symmetric metric must be of a special form.**

$$ds^2 = h(r,t)dt^2 + k(r,t)\left[d\theta^2 + \sin^2\theta d\phi^2\right] + l(r,t)dr^2 + 2a(r,t)drdt$$

Central symmetry means that there is an action of the rotation group $SO(3)$ on the co-ordinates that leaves the metrc invariant. The co-ordinates must split into the spatial ones x^a which transform as a 3-vector and time t which is a scalar. The radial co-ordinate $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ is again rotation invariant. Note that $dr = \frac{x^a}{r}dx^a$.

The metric tensor $g_{\mu\nu}$ can be split into a 3×3 symmetric matrix g_{ij}, a three-dimensional vector g_{0i}, and a scalar g_{00}. The scalar g_{00} must be a function of the radial co-ordinate and time alone. Central symmetry implies that any vector must be proportional to the positin vecetor: that the vector g_{0i} has only radial component. This means that $g_{0i}dx^i = a(r,t)dr$ for some scalar funtion of distance and time. The only independent centrally symmetric 3-tensors are δ_{ij} and $x_i x_j$. Thus tpatial metric must be of the form

$$g_{ij}dx^i dx^j = k(r,t)\left[d\theta^2 + \sin^2\theta\right]d\phi^2 + l(r,t)dr^2$$

We still have the freedom to make co-ordinate transformations that mix r, t as long as they don't involve the angular variables.

$$r \mapsto R(r,t), \quad t \mapsto T(r,t)$$

1.3.1. *We can choose a co-ordinate system such that* $k(r,t) = -r^2$. Under the above change of variables, $k(r,t)$ transforms as a scalar. We can simply choose the square root of $-k$ as the radial co-ordinate. The sign is negative because the metric has Lorentzian signature. We still have the freedom to make changes in the time variable $t \mapsto T(r,t)$.

1.3.2. *We can choose either* $g_{rt} = 0$ *or* $g_{rr} = -1$. These lead to two co-ordinate systems, each with some advantage and disadvantage. Schwarschild originally chose the system with $g_{rt} = 0$. Painleve later suggested that the choice $g_{rr} = -1$ leads to another system in which the structure of the event horizon is simpler to understand.

$$h(r,t)dt^2 + l(r,t)dr^2 + 2a(r,t)drdt \mapsto h\dot{T}^2dt^2 + dr^2\left[l + hT'^2 + 2aT'\right] + 2drdt\left[a\dot{T} + hT'\dot{T}\right]$$

So we solve either

$$a + hT' = 0$$

or

$$l + hT'^2 + 2aT' = -1.$$

We will for now make the first choice and follows Schwarschild's steps in solving the equations. But it will be useful to return later and re-express the solution in Painleve co-ordinates.

1.3.3. *We still have the freedom to make the transformation* $t \mapsto T(t)$ *that does not involve* r. This will come in handy later.

1.4. **By a choice of co-ordinates the metric can be brought to the diagonal form.**

$$ds^2 = e^\nu dt^2 - r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right] - e^\lambda dr^2$$

The change of notation $h = e^\nu, l = -e^\lambda$ will simplify some later formulas.

1.5. **The action principle for geodesics can be used to derive formulas for the Christoffel symbols.**

$$S = \frac{1}{2}\int \left[e^{\nu}\dot{t}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - e^{\lambda}\dot{r}^2\right] d\tau$$

Varying, the geodesic equations become, with $\left(x^0, x^1, x^2, x^3\right) = (t, r, \theta, \phi)$
The simplest variation is w.r.t. ϕ:

$$\frac{d}{d\tau}\left[r^2\sin^2\theta\dot{\phi}\right] = 0, \implies \ddot{\phi} + 2\frac{\dot{r}}{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0$$

from which

$$\Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^3_{23} = \cot\theta$$

The next simplest is variation w.r.t. θ:

$$\frac{d}{d\tau}\left[r^2\dot{\theta}\right] - r^2\sin\theta\cos\theta\dot{\phi}^2 = 0 \implies \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0$$

so that we can read off

$$\Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

Then we vary w.r.t. t (with $\lambda_t = \partial_t\lambda$ etc.)

$$\frac{d}{d\tau}\left[e^{\nu}\dot{t}\right] + \lambda_t e^{\lambda}\dot{r}^2 = 0$$

$$\ddot{t} + \nu_r\dot{r}\dot{t} + \nu_t\dot{t}^2 + \lambda_t e^{\lambda-\nu}\dot{r}^2 = 0$$

and

$$\Gamma^0_{01} = \frac{1}{2}\nu_r, \quad \Gamma^0_{00} = \nu_t, \quad \Gamma^0_{11} = \lambda_t e^{\lambda-\nu}$$

Finally we get the variation w,r,t. r

$$\frac{d}{d\tau}\left[-e^{\lambda}\dot{r}\right] - \nu' e^{\nu}\dot{t}^2 + r\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = 0$$

$$\ddot{r} + \lambda_r\dot{r}^2 + \lambda_t\dot{r}\dot{t} + \nu' e^{\nu-\lambda}\dot{t}^2 - re^{-\lambda}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = 0$$

so that

$$\Gamma^1_{11} = \lambda_r, \quad \Gamma^1_{10} = \frac{1}{2}\lambda_t, \quad \Gamma^1_{22} = -re^{-\lambda}, \quad \Gamma^1_{33} = -re^{-\lambda}\sin^2\theta$$

The remaining components of $\Gamma^{\mu}_{\nu\rho}$ are either permutations of the above or are zero.