GRAVITATION F10

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Lecture 17

1. The Schwarschild Blackhole

1.1. **The T-axis is time-like for $r > r_s$ and space-like for $r < r_s$.** The metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dT^2 - 2\sqrt{\frac{r_s}{r}} dr dT - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

is not diagonal. Still, if only T varies, its coefficient determines the nature of the T-axis. That sign flips as we cross the Schwarschild radius. Thus it is not correct anymore to think of T as time inside the Scwarschild radius: both T and $r$are spacelike co-ordinates. The metric in the rT plane $\begin{pmatrix} 1 - \frac{r_s}{r} & -\sqrt{\frac{r_s}{r}} \\ -\sqrt{\frac{r_s}{r}} & -1 \end{pmatrix}$ always has a positive and a negative eigenvalue because the determinant is -1. But these directions are some linear combinations of dr, dT not themselves.

1.2. **A radial light ray satisfies a simple ODE in Painleve co-ordinates.** The condition for a null geodesic with constant angles is

$$\left(1 - \frac{r_s}{r}\right) dT^2 - 2\sqrt{\frac{r_s}{r}} dr dT - dr^2 = 0$$

$$\left[\left(1 - \sqrt{\frac{r_s}{r}}\right) dT - dr\right] \left[\left(1 + \sqrt{\frac{r_s}{r}}\right) dT + dr\right] = 0$$

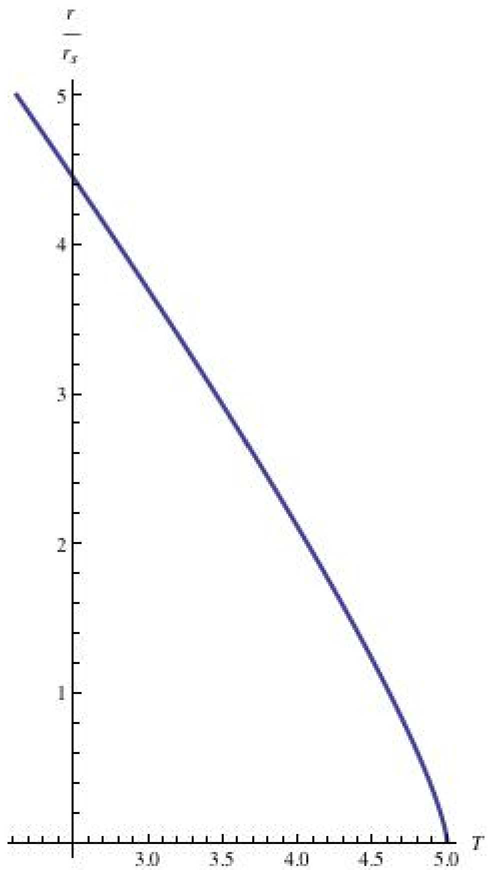
1.2.1. *One of the solutions describes the incoming light ray.*

$$\frac{dr}{dT} = -\left(1 + \sqrt{\frac{r_s}{r}}\right)$$

The solution is

$$T = T_0 + 2\sqrt{r_s r} - r - 2r_s \log\left[1 + \sqrt{\frac{r}{r_s}}\right]$$

This curve connects large values of r to small values monotonically. The incoming light ray falls into the center no matter where it starts.



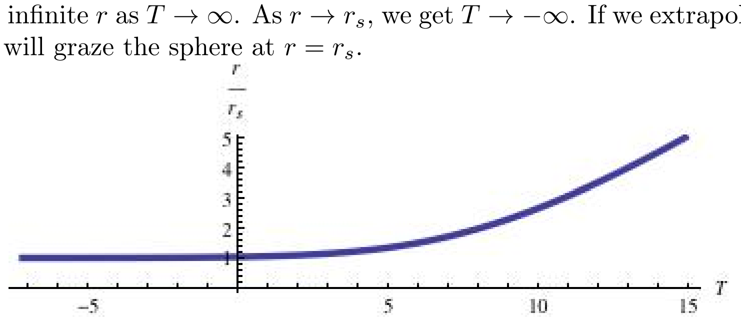
1.2.2. *The other solution is an outgoing ray when $r > r_s$.*

$$\frac{dr}{dT} = 1 - \sqrt{\frac{r_s}{r}}$$

The solution is

$$T = T_0 + 2\sqrt{r r_s} + r + 2r_s \log\left[-1 + \sqrt{\frac{r}{r_s}}\right]$$

If we start at a point outside the Schwarschild radius, this is a curve that escapes to infinite r as $T \to \infty$. As $r \to r_s$, we get $T \to -\infty$. If we extrapolate it backwards it will graze the sphere at $r = r_s$.



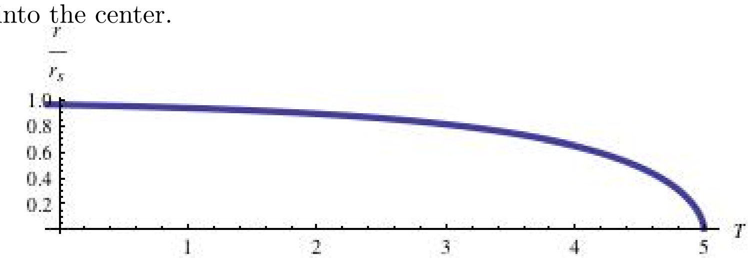
1.2.3. *When $r < r_s$ there is no solution that escapes to infinity.* The ray that should have been outgoing would have solved again

$$\frac{dr}{dT} = 1 - \sqrt{\frac{r_s}{r}}$$

When $r < r_s$ the solution is instead

$$T = T_0 + 2\sqrt{r r_s} + r + 2r_s \log\left[1 - \sqrt{\frac{r}{r_s}}\right]$$

The sign inside the square brackets is flipped to make the log real valued. For any value of T , the solution for r remains inside the Schwarschild radius. Thus if the ray starts out inside the Schwarschild radius, both solutions describe it falling into the center.



1.3. **No light ray inside the Schwarschild radius ($r < r_s$) can escape to infinity.** (To prove this completely, we must also consider the non-radial null geodesics and they also have decreasing r.)

1.4. **There are no causal paths that connect the interior of a blackhole with its exterior: the sphere $r = r_s$ is an event horizon.** A causal path is a future pointing time-like or null curves: these are the allowed paths for a particle of light or matter. Any such curve will turn around and head towards the origin as it approaches the Schwarschild radius. It does not have to be a geodesic for this to happen.This is strange, but no law of physics is violated by the existence of such a surface. It is called an event horizon.

1.5. **An symptotically flat metric with an event horizon is called a blackhole.** Asymptotically flat means that at larg distances the metric tends to Minkowski space. Faraway observers cannot see the inside the event horizon; not even light escapes it.

1.6. **A blackhole is a perfect absorber.** Everything that crosses the sphere $r = r_s$ stays inside. By further analysis we can show that not only light but any form of energy is absorbed into the center once it crosses the Schwarschild radius. For example, the wave equation has no solutions wth a source inside $r < r_s$ with energy escaping to infinity.

2. The Wave Equation In A Blackhole Background

Light is actually described by a wave equation; it is only in the limit of small wavelength that it can described as rays. It is useful to understand how waves propagate in a blackhole background. It is a little simpler to solve the scalar wave equation rather than Maxwell's equations.

2.1. The scalar wave equation in a blackhole background reduces to an ODE after separation of variables.

The wave equation is

$$\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\chi\right]=0$$

In Painleve co-ordinates the metric gives

$$\sqrt{-g}=r^2,\quad g^{\mu\nu}=\begin{pmatrix}1 & -\sqrt{\frac{r_s}{r}} & 0 & 0\\ -\sqrt{\frac{r_s}{r}} & \frac{r_s}{r}-1 & 0 & 0\\ 0 & 0 & -\frac{1}{r^2} & 0\\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta}\end{pmatrix}$$

$$\partial_T\left[r^2\partial_T\chi\right]-\partial_T\left[r^2\sqrt{\frac{r_s}{r}}\partial_r\chi\right]-\partial_r\left[r^2\sqrt{\frac{r_s}{r}}\partial_T\chi\right]+\partial_r\left[\left(\frac{r_s}{r}-1\right)\partial_r\chi\right]-\partial_\theta^2\chi-\partial_\phi\left[\frac{1}{\sin^2\theta}\partial_\phi\chi\right]=0$$

We recognize that the angular terms are the same as in flat space: so a solution is in terms of the usual spherical harmonics. Also, T never appears explicitly, so the dependence on it has to be exponential.

So we can separate variables:

$$\chi(T,r,\theta,\phi)=e^{i\omega T}R(r)Y_{lm}(\theta,\phi)$$

It is useful to think in terms of the variational principle

$$S=\frac{1}{2}\int\sqrt{-g}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi dx$$

which reduces to (we drop an overall multiplicative constant)

$$S=\frac{1}{2}\int_0^\infty\left[\left(-\omega^2+\frac{l(l+1)}{r^2}\right)R^2-2i\omega\sqrt{\frac{r_s}{r}}RR'+\left(\frac{r_s}{r}-1\right)R'^2\right]r^2dr$$

Integrating by parts the second term

$$S=\frac{1}{2}\int_0^\infty\left[\left(-r^2\omega^2+l(l+1)+\frac{3}{2}i\omega\sqrt{r_sr}\right)R^2+r^2\left(\frac{r_s}{r}-1\right)R'^2\right]dr$$

This leads to a Sturm-Liouville type ODE $\frac{d}{dr}\left[pR'\right]+qR=0$

$$\frac{d}{dr}\left[r(r-r_s)R'\right]+\left(-r^2\omega^2+l(l+1)+\frac{3}{2}i\omega\sqrt{r_sr}\right)R=0$$

The vansihing of p at $r=r_s$ can be used to show that the initial data in the interior $r<r_s$ has no effect on the exterior solution.. A solution in terms of elementary functions is not possible; but the nature of the solution can be determined by asymptotic analysis. Because the effective potential has an imaginary part, we can see that incoming waves are partially abosrbed and the rest is scattered back.

There are also ‘resonance’ solutions that are like bound states (vanishing at radial infinity) with complex eigenvalue for ω. These describe eigenstates that decay with time, as they are absorbed into the center. See the book on Blackholes by Chnadrashekhar for a more in depth analysis.