GRAVITATION F10

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Lecture 18

1. Lie Derivatives

1.1. **A vector field is an infinitesimal transformation of points.** Given a vector field, from each point x we can move to a nearby point x' in the direction determined by it:

$$x^\mu \mapsto x'^\mu \equiv x^\mu + \epsilon v^\mu(x)$$

A scalar field transforms as

$$\phi(x') \approx \phi(x) + \epsilon v^\mu \partial_\mu \phi$$

Thus the infinitesimal transformation of a scalar under a vector field is just

$$\mathcal{L}_v \phi = v^\mu \partial_\mu \phi.$$

This is called the *Lie derivative* of ϕ by v .

The infinitesimal transformation $\mathcal{L}_v \omega$ of a covariant vector field ω_μ can be found from the invariance of the combination $\omega_\mu dx^\mu$

$$\omega(x')dx'^\mu = \omega_\mu(x)dx^\mu + \epsilon \left[v^\nu \partial_\nu \omega_\mu dx^\mu + \omega_\mu \partial_\nu v^\mu dx^\nu\right]$$

so that its Lie derivative is

$$[\mathcal{L}_v \omega]_\mu = v^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu v^\nu$$

A covariant symmetric tensor (like the metric) has a Lie derivative that can be calculated similarly:

$$[\mathcal{L}_v g]_{\mu\nu} = v^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu v^\rho + g_{\mu\rho} \partial_\nu v^\rho$$

It is interesting to rewrite this in terms of the covariant form of the vector field $v_\mu = g_{\mu\rho} v^\rho$

$$[\mathcal{L}_v g]_{\mu\nu} = v^\rho \partial_\rho g_{\mu\nu} - v^\rho \partial_\mu g_{\rho\nu} - v^\rho \partial_\nu g_{\mu\rho} + \partial_\mu v_\nu + \partial_\nu v_\mu$$

Recalling the formula for the covariant derivative of a vector field we see that the Lie derivative of the metric can be written as

$$[\mathcal{L}_v g]_{\mu\nu} = D_\mu v_\nu + D_\nu v_\mu.$$

1.2. **If the Lie derivative of the metric by a vector field is zero, it is a Killing vector.** Killing vectors describe infinitesimal symmetries of the metric.

2. Killing Vectors

2.1. **Translations and rotations are the symmetries of the Euclidean metric.** Infinitesimally, a symmetry of the Euclidean metric is a vector field that leaves its metric unchanged under the transformation $x^i \mapsto x^i + \epsilon v^i$.

$$\partial_i v_j + \partial_j v_i = 0$$

A constant is a solution: translation. a linear function $v_i(x) = r_{ij} x^j$ is a solution if

$$r_{ij} + r_{ji} = 0$$

These are infinitesimal rotations. For example a rotation through an angle θ in the $x^1 x^2$plane has the effect

$$x^1 \mapsto \cos\theta x^1 + \sin\theta x^2$$

$$x^2 \mapsto -\sin\theta x^1 + \cos\theta x^2$$

If the angle is infinitesimally small

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \mapsto \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

Any anti-symmetrc tensor can be written as the sum of infinitesimal rotations in planes such as this.

There are no other solutions to the conditions above.

2.2. **Translations and Lorentz transformations are the symmetries of the Minkowski metric.** The Killing vectors satisfy

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 0$$

The solutions are

$$v_\mu = a_\mu + \lambda_{\mu\nu} x^\nu$$

where

$$\lambda_{\mu\nu} = -\lambda_{\nu\mu}$$

These ten independent solutions can be thought of as translations a_μ, rotations λ_{ij} and Lorentz boosts λ_{0i}.

2.3. **Killing vectors are infinitesimal symmetries of the geodesic variational principle.** Recall

$$S = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

Under an infinitesimal transformation $x^\mu \mapsto x^\mu + \epsilon v^\mu$ the change in the action is

$$\delta_v S = \frac{1}{2} \int \left[v^\rho \partial_\rho g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dv^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dv^\nu}{d\tau} \right] d\tau$$

$$= \frac{1}{2} \int \left[v^\rho \partial_\rho g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \partial_\rho v^\mu \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \partial_\rho v^\nu \frac{dx^\mu}{d\tau} \frac{dx^\rho}{d\tau} \right] d\tau$$

$$= \frac{1}{2} \int \left[\mathcal{L}_v g_{\mu\nu}\right] \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

Thus a Killing vector leads to a symmetry of the geodesic equation.

2.4. **To each Killing vector there corresponds a conserved quantity of the geodesic equation.** This is a special case of a much more general theorem of Noether: symmetries in a variational principle lead to conservaton laws.

We have

$$\frac{d}{d\tau}\left[g_{\mu\nu} v^\mu \frac{dx^\nu}{d\tau}\right] = v^\mu \frac{d}{d\tau}\left[g_{\mu\nu} \frac{dx^\nu}{d\tau}\right] + g_{\mu\nu} \partial_\rho v^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau}$$

For a geodesic

$$\frac{d}{d\tau}\left[g_{\mu\nu} \frac{dx^\nu}{d\tau}\right] = \frac{1}{2} \partial_\mu g_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau}$$

so that

$$\frac{d}{d\tau}\left[g_{\mu\nu} v^\mu \frac{dx^\nu}{d\tau}\right] = v^\mu \frac{1}{2} \partial_\mu g_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \partial_\rho v^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau}$$

$$\frac{d}{d\tau}\left[g_{\mu\nu} v^\mu \frac{dx^\nu}{d\tau}\right] = \frac{1}{2} \left[\mathcal{L}_v g\right]_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau}$$

Thus if v is a Killing vector field and x is a geodesic we have the conservation law

$$\frac{d}{d\tau}\left[g_{\mu\nu} v^\mu \frac{dx^\nu}{d\tau}\right] = 0.$$

2.5. **Since the tangent vector to the geodesic has constant length we always have one conservation law.** We don't even need a Killing vector for this. This follows from the fact that τ does not appear explictly in the action principle. In the language of mechanics, the Hamiltonian is conserved when the Lagrangian is independent of the evolution parameter (τin our case).

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = H = \text{constant}.$$

This parameter has the physical meaning of the square of the mass.

For a time-like geodesic this constant is positive: the mass is real and non-zero. It is convenient to choose the unit of τ that this is unity; in this case τ is proper time. For null geodesics H will vanish: the mass is zero. Space-like geodesics have $H < 0$ but they don't have a meaning as the path of any particle. They can be of mathematical interest, however.