GRAVITATION F10

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Lecture 19

1. Time-Like Geodesics of the Schwarschild Metric

To solve any mechanical problem we must exploit conservation laws. Often symmetries provide clues to these conservation laws. We will determine the time-like geodesics in the Schwarschild metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

1.1. **A time-like geodesic satisfies.**

$$\left(1 - \frac{r_s}{r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{r_s}{r}} - r^2 \left(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2\right) = 1$$

Here the dot denotes derivatives w.r.t. τ.

1.2. **Translations in t and rotations are symmetries of the Schwarschild metric.** The angular dependence is the same as for the Minkwoski metric. The invariance under translations in t is obvious

1.3. **Thus the energy and angular momentum of a particle moving this gravitational field are conserved.** The translation in Tgives the conservation of energy per unit mass

$$E = \left(1 - \frac{r_s}{r}\right) \dot{t}$$

Rotations in ϕ lead to the conservation of the third component of angular momentum per unit mass

$$L = r^2 \dot{\phi}.$$

This is an analogue of Kepler's law of areas.

The conservation of angular momentum, which is a 3-vector, implies also that the orbit lies in the plane normal to it.

1.3.1. *We can choose co-ordinates such that the geodesic lies in the plane $\theta = \frac{\pi}{2}$.* By looking at the second component of the geodesic equation

$$\frac{d}{d\tau}\left[r^2 \frac{d\theta}{d\tau}\right] = r^2 \sin\theta\cos\theta \left[\frac{d\phi}{d\tau}\right]^2$$

we can see that $\theta = \frac{\pi}{2}$ is a solution. We can rotate the co-ordinate system so that any plane passing through the center corresponds to $\theta = \frac{\pi}{2}$.Thus

$$\left(1-\frac{r_s}{r}\right)\dot{t}^2-\frac{\dot{r}^2}{1-\frac{r_s}{r}}-r^2\dot{\phi}^2=1$$

1.4. **To determine the shape of the orbit we must detemine r as a function of ϕ.** In the Newtonian limit these are conic sections: ellipse, parabola or hyperbola. Let $u=\frac{r_s}{r}$ .Then

$$\dot{r}=r'\dot{\phi}=\frac{r'}{r^2}L=-lu'.$$

Here prime denotes derivative w.r.t. ϕ. Also $l=\frac{L}{r_s}$. So

$$\frac{E^2}{1-u}-\frac{l^2u'^2}{1-u}-l^2u^2=1$$

1.5. **We get an ODE for the orbit.**

$$l^2u'^2=E^2-1+u-l^2u^2+l^2u^3$$

This is the Weierstrass equation, solved by the elliptic integral . Since we are interested in the case where the last term (which is the GR correction) is small a different strategy is more convenient. Differentiate the equation to eliminate the constants:

$$u''+u=\frac{1}{2l^2}+\frac{3}{2}u^2$$

1.5.1. *In the Newtonian approximation the orbit is periodic.* The Newtonian approximation is

$$u_0''+u_0=\frac{1}{2l^2}\implies$$

$$u_0=\frac{1}{2l^2}+B\sin\phi$$

for some constant of integration B.

1.5.2. *Recall the equation for an ellipse in polar co-ordinates.*

$$\frac{1}{r}=\frac{1}{b}+\frac{\epsilon}{b}\sin\phi$$

Here, ϵ is the eccentricity of the ellipse: if it is zero the equation is that of a circle of radius b. In general b is the semi-latus rectum of the ellipse. If $1>\epsilon>0$, the closest and farthest approach to the origin are at $\frac{1}{r_{1,2}}=\frac{1}{b}\pm\frac{\epsilon}{b}$ so that the major axis is $r_2+r_1=\frac{2b}{1-\epsilon^2}$. So now we know the meaning of l and B in terms of the Newtonian orbital parameters.

$$b=2r_sl^2,\quad B=\frac{\epsilon}{b}r_s$$

1.5.3. *We can find the GR correction to the orbit by perturbing around the Newtonian solution.* Putting

$$u = u_0 + u_1$$

to first order

$$u_1'' + u_1 = \frac{3}{2}u_0^2$$

$$= \frac{3}{8l^4} + \frac{3B}{2l^2}\sin\phi + \frac{3}{2}B^2\sin^2\phi$$

$$u_1'' + u_1 = \frac{3}{8l^4} + \frac{3}{4}B^2 + 3\frac{B}{2l^2}\sin\phi - \frac{3}{4}B^2\cos 2\phi$$

Although the driving terms are not periodic, the solution is not periodic,because of the resonant term $\sin\phi$ in the r.h.s.

$$u_1 = \text{periodic} + \text{constant}\phi\sin\phi$$

1.6. **In GR the orbit is not closed.** Thus GR predicts that as a planet returns to the perihelion its angle has suffered a net shift. After rewriting B, l, r_s,in terms of the parameters a, ϵ, T of the orbit, the perihelion shift is found to be

$$\frac{24\pi^2 a^2}{(1-\epsilon^2)c^2T^2}$$

where a is the semi-major axis and T is the period of the orbit.

1.7. **This perihelion shift agrees with the measured anomaly in the orbit of Mercury.** At the time Einstein proposed his theory, such a shift in the perihelion of Mercury was already known-and unexplained- for a hundred years! The prediction of GR, $43''$ of arc per century, exactly agreed with the observation: its first experimental test. For the Earth the shift of the perihelion is even smaller: $3.8''$ of arc per century. Much greater accuracy has been possible in determining the orbit of the Moon through laser ranging. The resuls are a quantitative vindication of GR to high precision.