GRAVITATION F10

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Lecture 21

1. Sources of Gravity

1.1. **The stress tensor is the source of gravity.** Recall that the source of electromagnetism is the electric current density. This is a four-vector whose time component is the charge density and the space components are the current density. It is conserved

$$\partial_\mu \left[\sqrt{-g} j^\mu\right] = 0 \implies \partial_0 \int j^0 \sqrt{-g} d^3x = 0$$

The analogous quantity for gravity is the stress tensor.(See Lect 12). It is a symmetric tensor which is also conserved

$$D_\mu T^{\mu\nu} = 0$$

The conserved quantity is now the vector representing momentum

$$P^\mu = \int T^{\mu 0} \sqrt{-g} d^3x$$

In particular

$$P^0 = \int T^{00} d^3x$$

is the total energy.

1.1.1. *We should expect from the Newtonian theory that mass density is a source for gravity.* But mass can be converted to energy. So it must be the total energy density that plays this role. Moreover, only if we combine energy and momentum do we get a Lorentz invariant quantity P^μ. Being a vector, its density is the zeroth component of a tensor.

1.2. **The stress tensor of matter is the variation of its action with respect to the metric.** More precisely

$$\sqrt{-g} T_{\mu\nu} = \frac{\delta S_m}{\delta g^{\mu\nu}}$$

Thus for a scalar field satisfying the wave equation

$$D^\mu \partial_\mu \phi = 0$$

the action of this matter field is

$$S_m = \frac{1}{2}\int \partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g}d^4x$$

Variations with respect to $g^{\mu\nu}$ gives

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}\partial_\rho\phi\partial_\sigma\phi g^{\rho\sigma}g_{\mu\nu}$$

The second term arises from the identity

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g^{\mu\nu}\sqrt{-g}.$$

The conservation of $T^{\mu\nu}$ follows from the wave equation

$$D^\mu T_{\mu\nu} = \frac{1}{2}[D^\mu\partial_\mu\phi]\partial_\nu\phi + \frac{1}{2}\partial_\mu\phi D^\mu\partial_\nu\phi - \frac{1}{4}\times 2D_\nu\partial_\rho\phi\partial_\sigma\phi g^{\rho\sigma}$$

The last two terms cancel if we use

$$D_\nu\partial_\rho\phi = D_\rho\partial_\nu\phi$$

which is true for scalars. The remaining term is zero by the wave equation.

1.2.1. *The stress tensor for the electromagnetic field also follows from its action.* We saw in Lect 12 that

$$T_{\mu\nu} = -\frac{1}{2}F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} + \frac{1}{8}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}$$

1.2.2. *The stress tensor of a fluid depends on its equation of state.* The stress tensor of a relativistic fluid is

$$T_{\mu\nu} = [\rho + p]u_\mu u_\nu - pg_{\mu\nu}$$

Here, u^μ is the velocity field of the fluid. Conservation of $T_{\mu\nu}$ implies the equation of motion for the fluid.

The equation of state of the matter in the fluid gives the pressure p as a function of the mass and energy density ρ. We are ignoring dissipative effects like viscosity. Also we are assuming the temperature of the fluid to be a constant.More realistic models of a relativistic fluid have been developed, which have important applications in astrophysics.

1.3. **The Einstein tensor is a divergenceless combination of Ricci tensor and Ricci scalar.** The Einsein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \quad R = R^\rho_\rho.$$

The Bianchi identity implies that

$$D^\mu R_{\mu\nu} = \frac{1}{2}D_\nu R.$$

All other divergenceless combinations are constant multiples of $G_{\mu\nu}$.

1.4. **The Einstein tensor is proportional to the stress tensor.** These are the equations that determine the gravitational field, analogous to the Maxwell equations for the electromagnetic field.

$$G_{\mu\nu} = kT_{\mu\nu}$$

By taking traces we can also write this as

$$R_{\mu\nu} = k[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T], \quad T = T^{\rho}_{\rho}.$$

1.4.1. *The constant of proportionality can be determined by comparison with the Newtonian theory.* It involves Newton's constant and c. Recall that in the Newtonian approximation

$$ds^2 \approx c^2dt^2 - dx_idx_i + 2\phi(x)dt^2$$

with $|\phi| << c^2$. It is independent of time for a static source. The inverse square law of the gravitational force implies the Poisson equation for the potential:

$$\partial_i\partial_i\phi = 4\pi G_N\rho$$

where G_N is Newton's constant. On the other hand the Ricci tensor is the newtonian approximation (time derivatives are lower order in c^{-1})

$$R_{tt} \approx \frac{1}{c^2}\partial_i\partial_i\phi$$

All other components are even smaller for large c. On the other hand, $T_{tt} \approx \rho c^2$ and all other components of the stress tensor are negligible. Thus

$$\frac{1}{c^2}\partial_i\partial_i\phi = \frac{1}{2}kc^2\rho$$

By comparsion we get

$$k = 8\pi\frac{G_N}{c^4}$$

Thus Einstein's equations do not depend on any new constant of nature: we need only the speed of light and the Newton's constant. But,

1.5. **It is possible to add a term proportional to the metric to the equation of motion: the cosmological constant.** The principle of General Covariance as well as the conservation of energy-momentum are still satisfied if the equation of motion were slightly modified:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu}$$

The cosmological constant Λ was believed to be zero for a long time. Recent measurements of acceleration of distant supernovas seem to indicate that it has a small positive value. It is called "Dark Energy" by popular writers of science.

2. The Hilbert Variational Principle

2.1. **Hilbert showed that Einstein's equations follow from a variational principle.** We might suspect that there is a scalar function built out of the metric and its derivatives

$$S = \int L[g, \partial g]\sqrt{-g}dx$$

such that

$$\frac{\delta S}{\delta g^{\mu\nu}} = \sqrt{-g}G_{\mu\nu}$$

The only obvious candidate is the Ricci scalar. We can show (see Landau-Lifshitz) that indeed,

$$\delta \int R\sqrt{-g}dx = \int G_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}dx$$

for variations $\delta g^{\mu\nu}$ that vanish on the boundary.Thus, we can regard the Einstein's equations as following from the variational principle

$$S = -\frac{1}{k}\int R\sqrt{-g}dx + S_m$$

where $S_m$is the action of the source (matter).

2.2. **The Hilbert action can be written in terms of the square of the first derivatives of the metric up to a boundary term.** The Ricci scalar involves second derivatives of the metric. Analogy with other field equations (scalar, Maxwell) suggest that the action should only depend on the metric and its first derivative. Why is there a discrepancy?

Explicitly

$$R = g^{\mu\rho}R^{\sigma}_{\mu\sigma\rho}$$

$$R^{\sigma}_{\mu\nu\rho} = \partial_\nu\Gamma^{\sigma}_{\mu\rho} - \partial_\rho\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\nu\alpha}\Gamma^{\alpha}_{\rho\mu} - \Gamma^{\sigma}_{\rho\alpha}\Gamma^{\alpha}_{\nu\mu}$$

so that

$$\sqrt{-g}R = \sqrt{-g}g^{\mu\rho}\left[\partial_\sigma\Gamma^{\sigma}_{\mu\rho} - \partial_\rho\Gamma^{\sigma}_{\mu\sigma} + \Gamma^{\sigma}_{\sigma\alpha}\Gamma^{\alpha}_{\rho\mu} - \Gamma^{\sigma}_{\rho\alpha}\Gamma^{\alpha}_{\sigma\mu}\right]$$

This can be written as

$$\sqrt{-g}R = \partial_\sigma\left\{\sqrt{-g}g^{\mu\rho}\Gamma^{\sigma}_{\mu\rho} - \sqrt{-g}g^{\mu\sigma}\Gamma^{\rho}_{\mu\rho}\right\}$$

$$-\partial_\sigma\left\{\sqrt{-g}g^{\mu\rho}\right\}\Gamma^{\sigma}_{\mu\rho} + \partial_\sigma\left\{\sqrt{-g}g^{\mu\sigma}\right\}\Gamma^{\rho}_{\mu\rho}$$

$$+\sqrt{-g}g^{\mu\rho}\left\{\Gamma^{\sigma}_{\sigma\alpha}\Gamma^{\alpha}_{\rho\mu} - \Gamma^{\sigma}_{\rho\alpha}\Gamma^{\alpha}_{\sigma\mu}\right\}$$

Now we see the point: $\sqrt{-g}R$ involves only the square of the first derivatives of g plus a total serivative. A total derivative can always be written as a surface integral (Gauss' theorem) and hence won't affect the variation of S: the variation $\delta g^{\mu\nu}$ is required to vanish at the boundary.

2.2.1. *The lagrangian of gravity can be expressed in terms of the Christoffel symbols using standard identities.* Here the point is that we can express any first derivative of the metric in terms of the Christoffel symbols:

$$\partial_\mu g^{\nu\rho} = -\Gamma^\nu_{\sigma\mu} g^{\sigma\rho} - \Gamma^\rho_{\sigma\mu} g^{\sigma\nu}$$

There are also identities like

$$\Gamma^\sigma_{\rho\sigma} = \frac{1}{\sqrt{-g}} \partial_\rho \sqrt{-g}$$

and

$$g^{\mu\rho}\Gamma^\sigma_{\mu\rho} = -\frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g} g^{\mu\sigma}\right]$$

which help. Therefore

$$-\partial_\sigma \left\{\sqrt{-g} g^{\mu\rho}\right\} \Gamma^\sigma_{\mu\rho} = 2\sqrt{-g}\Gamma^\mu_{\sigma\alpha} g^{\alpha\rho}\Gamma^\sigma_{\mu\rho} - \sqrt{-g}\Gamma^\alpha_{\sigma\alpha} g^{\mu\rho}\Gamma^\sigma_{\mu\rho}$$

$$\partial_\sigma \left\{\sqrt{-g} g^{\mu\sigma}\right\} \Gamma^\rho_{\mu\rho} = -\sqrt{-g}\Gamma^\mu_{\nu\alpha} g^{\nu\alpha}\Gamma^\rho_{\mu\rho}$$

$$\sqrt{-g}R = 2\sqrt{-g}\Gamma^\mu_{\sigma\alpha} g^{\alpha\rho}\Gamma^\sigma_{\mu\rho} - 2\sqrt{-g}\Gamma^\alpha_{\sigma\alpha} g^{\mu\rho}\Gamma^\sigma_{\mu\rho} + \sqrt{-g} g^{\mu\rho} \left\{\Gamma^\sigma_{\sigma\alpha}\Gamma^\alpha_{\rho\mu} - \Gamma^\sigma_{\rho\alpha}\Gamma^\alpha_{\sigma\mu}\right\}$$

and finally that

$$\int \sqrt{-g} R dx = \int \sqrt{-g} g^{\mu\nu} \left\{\Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma}\right\} dx + \text{boundary terms}$$

This is useful in calculating the Einstein's equations for various geometries (e.g., spherically symmetric).