GRAVITATION F10

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Lecture 22

1. The Raychaudhuri Equation

1.1. **Gravity is universally attractive.** This is what makes gravity so different from other forces of nature. There are many charged particles in the Sun, but we can ignore its electric field at long distances because the charges cancel. Not so for gravity: there are no negative masses. Gravity reigns supreme over all other forces at large distances.

1.2. **A collection of particles moving under the influence of each other's gravity will focus.** In the absence of other forces to balance it, gravity will cause a collapse.

1.3. **Raychaudhuri found a differential equation that helps study this collapse.** In essence, Raychaudhuri's equation is a theorem in geometry about the concergence of families of time-like geodesics. Let u^μ be a unit (time-like) vector field

$$g_{\mu\nu}u^\mu u^\nu = 1$$

We imagine that space-time is filled with matter, with the particle occupying position x moving with 4-velocty $u^\mu(x)$. At each location there is an instantaneous reference frame in which the particle at that position is at rest. The spatial components of a vector V^μ as judged by this co-moving observer are given by $h_{\mu\nu}V^\nu$ where

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu.$$

Recall that the infinitesimal change of the metric by a vector field is the Lie derivative

$$\mathcal{L}_u g_{\mu\nu} = D_\mu u_\nu + D_\nu u_\mu$$

The expansion tensor is the projection of this to the local spatial components

$$\theta_{\mu\nu} = \frac{1}{2}h_\mu^\rho h_\nu^\sigma \left[D_\rho u_\sigma + D_\sigma u_\rho\right]$$

(The factor of half is a convention.) The local change in spatial volume is its trace (the expansion scalar)

$$\theta = g^{\mu\nu}h_\mu^\rho h_\nu^\sigma D_\sigma u_\rho.$$

Using the fact that u^μ is a unite vector we can show that

$$\theta = D_\mu u^\mu.$$

If matter is expanding it is positive.

1.3.1. *Raychaudhuri derived a differential equation for this quantity:*

$$\dot{\theta} = \omega^2 - \sigma^2 - \frac{1}{3}\theta^2 - R_{\mu\nu}u^\mu u^\nu + D_\mu \dot{u}^\mu.$$

Here the dot denotes the covariant derivative along u^μ

$$\dot{X} = u^\mu D_\mu X$$

etc. Also,

$$\omega_{\mu\nu} = \frac{1}{2}h_\mu^\rho h_\nu^\sigma \left[D_\rho u_\sigma - D_\sigma u_\rho\right]$$

is the spatial projection of vorticity and

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3}h_{\mu\nu}\theta$$

is the part of the expansion that does not change the spatial volume.

1.3.2. *Proof of the Raychaudhuri equation.* It is easy enough to prove the equation once we know what we are looking for. We start with

$$D_\mu \dot{u}^\mu \equiv D_\mu \left[u^\nu D_\nu u^\mu\right]$$

$$= D_\mu u_\nu D^\nu u^\mu + u^\nu D_\mu D_\nu u^\mu$$

Now,

$$D_\mu D_\nu u^\mu = D_\nu [D_\mu u^\mu] + R^\mu_{\rho\mu\nu} u^\rho$$

so that

$$u^\nu D_\mu D_\nu u^\mu = \dot{\theta} + R_{\rho\nu}u^\rho u^\nu$$

by the definition of the Riemann and Ricci tensors and of θ.
Thus

$$\dot{\theta} = D_\mu \dot{u}^\mu - R_{\mu\nu}u^\mu u^\nu - D_\mu u_\nu D^\nu u^\mu$$

It remains to split the gradient of u into its symmetric traceless and anti-symmetric components to get the Raychaudhuri's equation.

$$D_\mu u_\nu = h_\mu^\rho h_\nu^\sigma D_\rho u_\sigma + u_\mu \dot{u}_\nu$$

$$= \omega_{\mu\nu} + \theta_{\mu\nu} + u_\mu \dot{u}_\nu$$

$$= \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}h_{\mu\nu}\theta + u_\mu \dot{u}_\nu$$

This is an orthogonal decomposition; that is, the trace of the product of any two distinct terms is zero. The last term has zero square as

$$u_\mu \dot{u}_\nu u^\nu \dot{u}^\mu = 0$$

for unit vectors. Also

$$\omega_{\mu\nu}\omega^{\nu\mu} = -\omega_{\mu\nu}\omega^{\mu\nu} \equiv -\omega^2$$

by anti-symmetry. Moreover

$$h_{\mu\nu}h^{\mu\nu} = 3$$

because it projects to a three dimensional space. (Also directly follows from its definition.) Putting all this together

$$D_\mu u_\nu D^\nu u^\mu = -\omega^2 + \sigma^2 + \frac{1}{3}\theta^2$$

from which the Raychaudhuri equation follows.

1.4. **Freely falling dust will collapse into a singularity.** Consider a very simple model of matter. (Too simple to be realsitic, actually) Each particle follows a geodesic so that $\dot{u} = 0$; the density is not strong enough to cause a gravitational field, so that $R_{\mu\nu} = 0$; and the flow is irrotational, so that $\omega = 0$. In this simplest case we get the differential inequality

$$\dot{\theta} \leq -\frac{1}{3}\theta^2$$

or

$$\frac{d}{d\tau}\left[\theta^{-1}\right] \geq \frac{1}{3}$$

This integrate to an inequality

$$\theta^{-1} \geq \theta_0^{-1} + \frac{\tau}{3}.$$

Thus once $\theta < 0$ at any point, somewhere along its future the particle will encounter a point where the expansion is negative infinity . Although this particular singularity is no big deal, Penrose and Hawking showed that a refined argument leads to devastating singularities in GR. A key idea is an inequality satisfied by the Ricci tensor that follows from the condition that energy density be positive.

2. The Energy Conditions

2.1. **The stress tensor has to satisfy cetrain inequalities in order that energy density be positive.** More precisely, the energy-momentum contained in any small volume must be either null or time-like; and it must be future pointing. This implies that

$$T_{\mu\nu}u^\mu u^\nu \geq 0$$

for any time-like vector field u. Through Einstein's equations, this implies that

$$G_{\mu\nu}u^\mu u^\nu \geq 0.$$

Now, what appears in geometry is the Ricci tensor. We should see what inequality the Ricci tensor satisfies.