GRAVITATION F10

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Lecture 3

1. Maxwell's Equations

Read the book by Jackson on *Classical Electrodynamics.* Or the second volume of the series by Landau and Lifshitz *Classical Theory of Fields.*

1.1. **All magnetic fields must have zero divergence.**

$$\nabla \cdot \mathbf{B} = 0$$

This means in particular that there is no analogue to an isolated electric charge in magnetism: a permanent magnet has to be a dipole. If you cut a dipole into two we will not get an isolated North pole and South pole. Instead we will get two dipoles again. Some theories that go beyond the standard model do allow for magnetic monopoles; but none have yet been observed.

1.2. **This equation can be solved by postulating that the magnetic field is a curl of a vector potential.**

$$\mathbf{B} = \nabla \times \mathbf{A}$$

1.3. **Two vector potentials that differ only by the gradient of a scalar give the same magnetic field.** This is called a gauge transformation

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \quad \mathbf{B}' = \mathbf{B}$$

$$\nabla \times \nabla\Lambda = 0.$$

It turns out that invariance under this transfomation is a fundamental symmetry of nature. Gauge transformations that generalize this are the fundamental symmetries of the standard model of elementary particles.

1.4. **Another equation of Maxwell relates the time derivative of the magnetic field to the eletric field.**

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$$

1.5. **We can solve this by postulating in addition a scalar potential V.**

$$\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla} V$$

Remark 1. Recall that we are using units such that $c = 1$. Otherwise there will be some factors of c all over the place.

The gauge transformations must now change the scalar potential as well

$$V' = V + \frac{\partial \Lambda}{\partial t}$$

so that the electric field is unchanged.

$$\frac{\partial \boldsymbol{\nabla} \Lambda}{\partial t} = \boldsymbol{\nabla} \frac{\partial \Lambda}{\partial t}.$$

1.6. **Under Lorentz transformations the scalar and vector potentials combine into a four-vector $A = (V, \mathbf{A})$.** We will introduce an index $\mu = 0, 1, 2, 3$ such that

$$A_0 = V, \quad A = (A_0, A_1, A_2, A_3)$$

Then the gauge transformation can be written as

$$A'_\mu = A_\mu + \partial_\mu \Lambda$$

where ∂_μ denotes differentiation along the μ th direction. Gauge invariance is based on the identity

$$\partial_\mu \partial_\nu \Lambda = \partial_\nu \partial_\mu \Lambda.$$

The electric and magnetic fields are then

$$E_i = \partial_0 A_i - \partial_i A_0, \quad i = 1, 2, 3.$$

$$B_1 = \partial_2 A_3 - \partial_3 A_2, \quad B_2 = \partial_3 A_1 - \partial_1 A_3, \quad B_3 = \partial_1 A_2 - \partial_2 A_1$$

This suggests that we combine them into a single matrix $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

It is an anti-symmetric matrix:

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

1.7. **Tensor index notation is a useful technical tool.** Recall that there is also a symmetric matrix $\eta^{\mu\nu}$ that allows us to take products of vectors. Its indices are written above as a way of keeping track of them:

$$\eta^{\mu\nu}p_\mu q_\nu = p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3, \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is also useful to think of the combination

$$\eta^{\mu\nu}p_\mu = p^\nu, \quad p^\nu = (p_0, -p_1, -p_2, -p_3)$$

as a vector with indices above and the scalar product as

$$p^\nu q_\nu = p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3$$

A pair of indices that are repeated are summed over.

$$p^\nu q_\nu = \sum_\nu p^\nu q_\nu.$$

This convention due to Einstein simplifies the appearance of equations. It means that you must be careful not to use the same index more than twice.

1.8. **The remaining Maxwell's equations can be written in Lorentz invariant form as.**

$$\partial^\mu F_{\mu\nu} = j_\nu$$

Expanded in terms of three dimensional quantities

$$\frac{\partial \mathbf{E}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{B} + \mathbf{j}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = j_0$$

The scalar j_0 is proportional to charge density and the vector $\mathbf{j}$ to current density.

1.9. **The potential A satisfies a wave equation.**

1.10. **The electromagnetic field describes a particle of mass zero and spin one.** Mass zero because it travels at the velocity of light. (Duh. it is light.) Spin one because in three dimensional language it includes a vector field, which has spin one.

1.11. **The equation of motion of a particle moving in an electromagnetic field is.**

$$\frac{du_\mu}{ds} = \frac{q}{m}F_{\mu\nu}u^\nu, \quad u^\mu = \frac{dx^\mu}{ds}.$$

Here, s is proper-time and u^μ the four velocity. When the velocity of the particle is small compared to that of light, $s \approx x^0 \approx t$. Also,

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{q}{m}\mathbf{E} + \frac{q}{m}\mathbf{v} \times \mathbf{B}$$

which is the Lorentz force equation.

Problem 2. Solve the equations of motion and prove the following two statements:

1.12. **In a constant magnetic field particle moves in a helix in space-time.**

1.13. **In a constant electric field the particle moves along a hyperbola in space-time.**