GRAVITATION F10

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Lecture 6

1. General Co-ordinates

1.1. **The choice of co-ordinate system should be adapted to the system being studied.** For example, curvilinear co-ordinates are useful in solving the Laplace equation in various geometries.

1.2. **The transformation between co-ordinates must be smooth functions.** Within a region where both co-ordinate systems are valid, the transformation between them must be differentiable and invertible. A simple example is the transformation between cartesion and polar co-ordinate systems. More generally, the new co-ordinate system x'^{μ} is specified as a set of functions of the old co-ordinates x^{μ} .

1.3. **The gradient of a scalar field transforms as.**

$$\frac{\partial f}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial f}{\partial x^{\nu}}$$

This is the chain rule of differentiation. Notice that the index ν is summed over. Another way to understand this transformation law is that the infinitesimal change in the scalar field is the same in both co-ordinate systems:

$$df = dx^{\mu}\frac{\partial f}{\partial x^{\mu}} = dx'^{\mu}\frac{\partial f}{\partial x'^{\mu}}$$

1.4. **If the derivatives of a function vanish at a point in one co-ordinate system, they vanish in any co-ordinate system.** The derivatives along the different co-ordinate axes of a function can be thought of as the components of a vector field. More generally,

1.5. **The components of a vector field change under co-ordinate transformations in a similar way:**

$$\omega'_{\mu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\omega_{\nu}.$$

1.5.1. *More precisely, fields that transform this way are called covariant vector fields. We will soon see another kind of vector called a contravariant (contra-gradient) vector that transforms oppositely.*

1.5.2. *Not every covariant vector field is the derivative of a function: the integrability condition is.*

$$\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = 0$$

Problem 1. Show that the above integrability condition is independent of changes of co-ordinates.

1.6. **The second derivatives of a function do not transform as a tensor.** More precisely, the second dervatives might vanish in at some point in one system but not in another.

By repeated use of the chain rule of differentiation,

$$\frac{\partial^2 f}{\partial x'^\rho \partial x'^\mu} = \frac{\partial x'^\sigma}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial^2 f}{\partial x^\sigma \partial x^\nu} + \frac{\partial x'^\sigma}{\partial x^\rho} \frac{\partial^2 x'^\nu}{\partial x^\mu \partial x^\sigma} \frac{\partial f}{\partial x^\nu}.$$

If the change of co-ordinates is linear, the last term vanishes; in general it won't be zero. Thus we will need some new notion of derivative to go beyond the fisrt derivative of a function.

1.7. **The tangent vector to a curve transforms as a contravariant vector: opposite to the gradient of a function.** A curve is given by specifying the co-ordinates as a function of some parameter $x^\mu(\tau)$. The components of the tangent vector are $\frac{dx^\mu}{d\tau}$.If we transform to some new co-ordinates

$$\frac{dx'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau}.$$

More generally, a vector whose components that transform this way is called a contravariant vector:

$$v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} v_\nu$$

Remember that the derivatives of x w.r.t. x' form the inverse matrix to the derivative of x' w.r.t. x.

$$\frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\rho.$$

Thus contravariant and covariant vectors transform opposite to each other.

1.7.1. *The Kronecker delta are components of the identity matrix.*

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

1.8. **The sum of the products of corresponding components of a covariant vector and a contravariant vector is a scalar: unchanged under co-ordinate transformations.**

$$\omega'_\mu = \frac{\partial x'^\nu}{\partial x^\mu}\omega_\nu, \quad v'^\mu = \frac{\partial x'^\mu}{\partial x^\rho} v_\rho$$

$$\implies \omega'_\mu v'^\mu = \omega_\nu \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial x'^\mu}{\partial x^\rho} v^\rho = \omega_\nu \delta^\nu_\rho v^\rho = \omega_\nu v^\nu.$$

We took care that an index appears at most twice in a factor. Also, a pair of repeated indices can be replaced by another without changing the value:

$$\omega_\nu v^\nu = \omega_\mu v^\mu.$$

2. The Metric in Curvilinear Co-Ordinates

2.1. **The infintesimal distance ds between two neighboring points in Euclidean space in Cartesian co-ordinates is given by.**

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu$$

2.2. **In a general co-ordinate system.**

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where $g_{\mu\nu}$ can depend on x^μ.

2.3. **The components transform under changes of co-ordinates as.**

$$g'_{\rho\sigma} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\mu} g_{\mu\nu}$$

To see this we have to remember that ds itself is independent of the co-ordinate system; and use the rule for each factor $dx'^\mu = \frac{\partial x'^\mu}{\partial x^\rho} dx^\rho$.

2.3.1. *We say that $g_{\mu\nu}$ are the components of the metric tensor.* Metric refers here to a measure of distance.

2.4. **We can calculate $g_{\mu\nu}$ by transforming from the Cartesian co-ordinate system or by some more direct geometrical argument.**

- In the polar co-ordinate system of the plane $ds^2 = dr^2 + r^2 d\phi^2$
- If we define $x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}}$, in Minkowski space $ds^2 = 2dx^+dx^-$. In this case the metric tensor is not diagonal.
- The metrix of R^3 in spherical polar co-ordinates is

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- For a more perverse example we note the prolate spheroidal coordinates in Euclidean space R^3 which is useful in solving some differential equations:

$$x^1 = \sinh r \sin\theta \cos\phi, \quad x^2 = \sinh r \sin\theta \sin\phi, \quad x^3 = \cosh r \cos\theta$$

$$ds^2 = \left[\sinh^2 r + \sin^2\theta\right]\left[dr^2 + d\theta^2\right] + \sinh^2 r \sin^2\theta d\phi^2$$

- More examples can be found in the monograph of Morse and Feshbach [1]. Or more conveniently on wikipedia these days.

REFERENCES

[1] Morse PM, Feshbach H (1953). Methods of Theoretical Physics, Part I. New York: McGraw-Hill