GRAVITATION F10

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Lecture 7

1. The Sphere

1.1. **The geometry of the sphere was studied by the ancients.** There were two spheres of interest to astronomers: the surface of the Earth and the celestial sphere, upon which we see the stars. Eratosthenes (3rd century BC) is said to have invented the use of the latitude and longitude as co-ordinates on the sphere. The (6th century AD) Sanskrit treatise *Aryabhatiya*, uses this co-ordinate system for the sphere as well (with the city of Ujjaini on the prime meridian) in solving several problems of spherical geometry. Predicting sunrise and sunset times, eclipses, calculating time based on the length of the shadow of a rod, making tables of positions of stars, are all intricate geometric problems.

1.2. **The metric of a sphere S^2 in polar co-ordinates is**

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

. We just have to hold r constant in the expression for distance in R^3 in polar co-ordinates.

1.3. **The sphere was the first example of a curved space.** There are no straightlines on a sphere: any straightline of R^3 starting at a point in S^2 will leave it. There are other subspaces of R^3 such as the cylinder or the cone which contain some straightlines. The question arises: what is the shortest line that connects two points on the sphere? Such questions were of much interest to map makers of the nineteenth century, an era when the whole globe was being explored. In the mid nineteenth century Gauss took up the study of the geometry of distances on curved surfaces metrics which was later generalized by Riemann to higher dimensions. Einstein realized a variant of Riemannian geometry, allowing for $ds^2$to be negative or zero as well, is the basis of a relativistic theory of gravity. For technical reasons, we will study a slightly different function than the length of a curve.

1.4. **The action of a curve on the sphere is defined to be.**

$$S = \frac{1}{2}\int \left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] d\tau$$

Note that this is not quite the same thing as the length of the curve:

$$l = \int \left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right]^{\frac{1}{2}} d\tau$$

It turns out that S is a simpler function on the space of curves than l . This is similar to the fact that x^2 is a differentiable function while $|x|$ is not.(Its derivative has a jump discontinuity at the origin.) But the same curves minimize S and l . (Again, both x^2 and $|x|$ are minimized at $x = 0$.)

1.4.1. *Some mathematicians, making a confused analogy with mechanics, call S the 'energy' of the curve instead of its action.*

1.5. **A geodesic is an extremum of the action.** This definition of a geodesic does not require it to be a minimum of the action or of distance: in fact many interesting geodesics are saddle points of S .

1.6. **The Euler-Lagrange equations of this variational principle give.**

$$\delta S = \int \left[\dot\theta\delta\dot\theta + \sin\theta\cos\theta\dot\phi^2\delta\theta + \sin^2\theta\dot\phi\delta\dot\phi\right] d\tau$$

$$-\ddot\theta + \sin\theta\cos\theta\dot\phi^2 = 0$$

$$\frac{d}{d\tau}\left[\sin^2\theta\dot\phi\right] = 0$$

1.6.1. *The key to solving any system of ODEs is to identify conserved quantities.* The obvious conserved quantity is

$$L = \sin^2\theta\dot\phi$$

1.6.2. *The solution is simplest when $L = 0$.* For these geodesics, $\phi$is a constant. Then θ is a linear function of τ. These are the lines of meridian of constant longitude. They are also called great circles. Geometrically they are the intersection of a plane passing through the center of the circle with the circle itself.

1.7. **Any pair of points on the circle lie on such a great circle. Thus geodesics are the same as arcs of great circles.** Using the symmetry of the sphere under rotations, we can always choose a co-ordinate system such that the two points lie along a longitude. So we don't actually have to solve the differential equations to see this fact. But if we have to find the equation of a geodesic with a given choice of axes,

1.7.1. *It is possible to solve the equations for any value of L.*

$$-\ddot\theta + \frac{\cos\theta L^2}{\sin^3\theta} = 0$$

Multiply by $\dot\theta$ and integrate once to get

$$\frac{1}{2}\dot\theta^2 + \frac{L^2}{2\sin^2\theta} = E$$

another constant of motion. Solving

$$\dot\theta = \sqrt{2E - \frac{L^2}{\sin^2\theta}}$$

$$\tau = \int \frac{d\theta}{\sqrt{2E - \frac{L^2}{\sin^2\theta}}}$$

which can be evaluated in terms of trig functions.

1.8. **The equator is a geodesic.**

1.9. **We can form a triangle with geodesics as sides, all of whose interior angles are right angles.** Start at the North Pole; go down to the equator along a meridian; go along the equator for a quarter of the circumference; then move along the meridian back to the North Pole.

1.10. **In Euclidean geometry, the sum of the interior angles must be π. In spherical geometry, it depends on the area enclosed by the sides.** A small geodesic triangle will have angles adding up to π as in Euclidean geometry. For small distances, geodesics appear to be straightlines and the sphere looks flat. This is why people thought the Earth was flat in olden days.

1.11. **Gauss found the correct measure of the curvature of a surface whose metric is given.**

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

1.12. **The sphere can also be idnetified with the complex plane, with the point at infinity added.** Indetofy the complex plane with the tangent plane to the sphere at the South plane. Given a point on the sphere, we can draw a straight line in R^3 that connects the North pole to that line: continuing that line, we get a point on the complex plane. This is the co-ordinate of the point. Thus the South pole is at the origin and the North point corresponds to infinity.

1.12.1. *The metric of S^2 is.*

$$ds^2 = 4\frac{d\bar{z}dz}{(1+\bar{z}z)^2}, \quad z = \tan\frac{\theta}{2}e^{i\phi}$$

1.12.2. *The isometries of the sphere are fractional linear transformations by $SU(2)$.*

$$z \mapsto \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Problem 1. Verify by direct calculations that these leave the metric unchanged.

This is one way of seeing that $SU(2)/\{1,-1\}$ is the group of rotations.

2. Hyperbolic Space

2.1. **The metric of hyperbolic geometry is.**

$$ds^2 = d\theta^2 + \sinh^2\theta d\phi^2$$

It describes a space of negative curvature. What this means is that two geodesics that start at the same point in slightly different directions will move apart at a rate faster than in Euclidean space. On a sphere, they move apart slower than in Euclidean space so it has positive curvature. Just as the sphere is the set of points at a unite distance from the center, 2.2. **The hyperboloid is the set of unit time-like vectors in Minkowski geometry $R^{1,2}$.** There is the co-ordinate system analogous to the spherical polar co-ordinate system valid in the time-like interior of the light come:

$$(x^0)^2 - (x^1)^2 - (x^2)^2 = \tau, \quad x^0 = \tau \cosh\theta,\ x^1 = \tau \sinh\theta \cos\phi,\ x^2 = \tau \sinh\theta \sin\phi$$

The Minkowski metric becomes

$$ds^2 = d\tau^2 - \tau^2 \left[d\theta^2 + \sinh^2\theta d\phi^2\right]$$

Thus the metric induced on the unit hyperboloid

$$(x^0)^2 - (x^1)^2 - (x^2)^2 = \tau,$$

is the one above.

2.3. **The hyperboloid can also be thought of as the upper half plane with the metric.**

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad y > 0$$

2.3.1. *The isometries are fractional linear transformations with real parameters a, b, c, d:*

$$z \mapsto \frac{az+b}{cz+d}, \quad ad - bc = 1$$

Problem 2. Verify that these are symmetries of the metric.

2.4. **The geodesics are circles orthogonal to the real line.** If two points have the same value of x, the geodesic is just the line parallel to the imaginary axis that contains them. Using the isometry above we can bring any pair of points to this configuration. It is also possible to solve the geodesic equations to see this fact.

2.5. **The hyperboloid can also be thought of as the interior of the unit disk.**

$$ds^2 = \frac{dz d\bar{z}}{(1 - \bar{z}z)^2}, \quad \bar{z}z < 1$$

Problem 3. What are the geodesics in this description?