PHY 458 Geometric Methods in Fluid Mechanics

S. G. Rajeev

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Mon/Wed at 10:25-11:40am in Meliora 219

First Class on Mon, Sep 8 2014

I will be using Blackboard to post Notes and Problems.

Synopsis

This course describes several ways in which geometrical methods can be applied to fluid mechanics. At an elementary level, it involves visualizing advection in terms of integral curves of vector fields. Or using curvilinear co-ordinates to solve Euler and Navier-Stokes equations in various geometries. Much of the material is old, although not usually presented geometrically. Some new ideas,speculations and directions of research will be presented on each topic.

The deepest contribution to the subject to date is by V.I. Arnold: his realization of Euler equations as geodesic equations of the diffeomorphism group and the computation of its sectional curvature.Also fundamental is the work of Holm, Marsden and Ratiu on the hamiltonian formulation of Euler-Poincare equations.

That this is negative in most directions explains the instability common in fluid mechanics. Along the way, we will touch on Smale's work on chaos; Aref's remarkably simple model of chaotic advection; how the topology of braids can be used to design mixing machines (e.g., for pizza dough) in mechanical engineering; and delve deep into the geometry of geodesics on a Lie group.

Pre-requisites

A basic knowledge of Riemannian geometry (e.g., a course on General Relativity AST231 or PHY413) and of fluid mechanics (e.g., PHY457). will be helpful. However, I will review what is needed and build the course from the ground up.

References

There is no one textbook that covers this ground. A mix of books and review articles will be used instead, along with my own notes.

Books

- V. I. Arnold and B. A. Khesin *Topological Methods in Hydrodynamics* Springer (1999)
- S. G. Rajeev *Advanced Mechanics From Euler's Determinism to Arnold's Chaos* Oxford(2013)

Journal Articles

- H. Aref *Stirring by Chaotic Advection* J. Fluid Mech. **143,** 1-21 (1984)
- S. Smale *Differentiable Dynamical Systems* Bull Amer Math Soc **73**, 747-817 (1967)
- V. I. Arnold Ann. Inst. Poly. Grenoble **16,** 319-361 (1966)
- J. Milnor *Curvatures of the Left Invariant Metrics on Lie Groups* Adv. Math. **29,** 293-329 (1976)
- D. D. Holm, J. E. Marsden and T. S. Ratiu, Adv. in Math., 137 (1998) 1-81, http://xxx.lanl.gov/abs/chao-dyn/9801015.

Online Articles

- T. Tao **What's New** *The Euler-Arnold equation* http://terrytao.wordpress.com/2010/06/07/the-euler-arnold-equation/
- S. G. Rajeev *Geometry of the Motion of Ideal Fluids and Rigid Bodies* http://arxiv.org/pdf/0906.0184v1.pdf
- J-L Thiffeault and M. D. Finn Topology, *Braids, and Mixing in Fluids* http://arxiv.org/pdf/nlin/0603003.pdf

Self-study

All learning is self-teaching. I will give exercises (two or three every other week) which will help you understand the material. Access to Mathematica or other symbolic/visualization software will be helpful. in solving them. To understand the field at a deeper level, you need to work on a reading/research projectof your own. I will suggest topics during the second half of the course. There will be no examinations.

Syllabus

Vector Fields

Vector Fields. Fixed points. Stable and Unstable Manifolds. Integral curve=advection. Foliations.

Homoclinic Points

Intersection of stable and unstable manifolds. Anosov maps. The Horse Shoe map of Smale.

Chaotic Advection

Aref model. Topological entropy. Braids. Models of mixing.

Curvilinear co-ordinates

Stream Function. Laplace equation. Dirichlet form. Metric and volume in curvilinear co-ordinates. Separable co-ordinates. Stackel matrix. Oblate and prolate spheroidal co-ordinates. Jacobi ellipsoidal co-ordinates. Ideal irrotational flow around a Frisbee or a Zeppelin.

Euler Equation

Ideal Incompressible Fluid. Hamiltonian formulation of Euler equations. Comparison to rigid body.

Lie Algebras and Groups

Commutation relations. Exponential map. Left invariant metric. Geodesic equation. Deduction of Euler and rigid body equations.

Riemannian Geometry

Metric. Geodesic. Sectional curvature as a biquadratic form. Examples: Hyperboloids, intersections of hyperboloids, separable and chaotic dynamics of geodesics. Negative sectional curvature $\Longrightarrow$ instability.

The Curvature of a Metric Lie Algebra

Computation of the curvature of a metric Lie algebra. Special case of the rigid body. Arnold's calculation of the curvature of Euler equations.

Dissipative Dynamics

Stokes flow. Dissipation tensor. Kirchoff laws and the resistance metric. How dissipation modifies geodesic dynamics. Special case of the Navier-Stokes equation.

Stochastic Dynamics

Langevin Equation. Fokker Plank equation. Effective Hamiltonian dynamics in the limit of small randomness.

Statistical Models of Turbulence

Kolmogorov scaling. Correlation Functions.