Asymptotic Methods in Science and Engineering
Problem Sets

March 21, 2012

1 Problem Set 1(Due Feb 15 2012)

1.1 Bessel Function

Findthe first two terms in the asymptotic expansion for large positive x for the Bessel function

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t} \cosh \nu t \; dt$$

1.2 Laplace's Method

Write a symbolic program in Mathemtica, Maple or Sage that implements Laplace's method for the integral

$$\int_a^b e^{\frac{1}{g}S(\phi)} f(\phi) d\phi$$

given functions S and f and the desired order in g. Verify by comparison with the expansion for the Bessel function.

1.3 Weak and Strong Coupling Expansion

Define

$$F(g) = \int e^{-\frac{1}{2}\phi^2 - g\phi^4} \frac{d\phi}{\sqrt{2\pi}}$$

Find the coefficients of the expansion in powers of g. Does it converge? By a change of variables obtain an expansion in powers of $g^{-\frac{1}{2}}$. Does it converge? By comparison of the first few terms with the numerical evaluation of the integral, determine the region in which each expansion is useful.

2 Problem set 2(Due Feb 29 2012)

2.1 Stieltjes Function

Define$f(J) = \int_0^\infty \frac{1}{1+\phi J} e^{-\phi} d\phi$. Show that for large J,

$$f(J) \approx \frac{\log J}{J}.$$

Using Pade' approximants to its asymptotic expansion in powers of J

$$f(J) \sim \sum_{n=0}^{\infty} (-1)^n n! J^n$$

derive rational approximations for $f(J)$ valid in the range $0 < J < 10$ to an accuracy of 1%. Verify this by numerical evalutaion of the integral.

Solution

Put $t = \frac{1+\phi J}{J}$ to get

$$\int_0^\infty \frac{1}{1+\phi J} e^{-\phi} d\phi$$

$$= \frac{e^{\frac{1}{J}}}{J} \int_{\frac{1}{J}}^\infty t^{-1} e^{-t} dt$$

$$= \frac{e^{\frac{1}{J}}}{J} \left[\int_{\frac{1}{J}}^1 t^{-1} e^{-t} dt + \int_1^\infty t^{-1} e^{-t} dt \right]$$

$$= \frac{e^{\frac{1}{J}}}{J} \left[\int_{\frac{1}{J}}^1 t^{-1} dt + \int_{\frac{1}{J}}^1 t^{-1} \left\{ e^{-t} - 1 \right\} dt + \int_1^\infty t^{-1} e^{-t} dt \right]$$

$$= \frac{e^{\frac{1}{J}}}{J} \left[\int_{\frac{1}{J}}^1 t^{-1} dt - \int_0^{\frac{1}{J}} t^{-1} \left\{ e^{-t} - 1 \right\} dt + \int_0^1 t^{-1} \left\{ e^{-t} - 1 \right\} dt + \int_1^\infty t^{-1} e^{-t} dt \right]$$

$$= \frac{e^{\frac{1}{J}}}{J} \left[\log J + \int_0^1 t^{-1} \left\{ e^{-t} - 1 \right\} dt + \int_1^\infty t^{-1} e^{-t} d + \mathrm{O}\left(\frac{1}{J} \right) \right]$$

So we should expect that the exact answer lies in between the Pade' approximants $P_N^N(J) \sim J^0$ and $P_{N+1}^N(J) \sim \frac{1}{J}$. Numerical evaluation of these will determine the value of N.

2.2 Tridiagonal and Cyclic Matrices

- Let $T_{kl} = 1$ if $|k - l| = 1$, and zero otherwise, for $k, l = 1, 2, \cdots M$. Find its eigenvalues for $M = 2, 3$ as well as in the limit as $M \to \infty$.
- What is the spectrum of T if we impose periodic boundary condition that $T_{M,1} = T_{1,M} = 1$? Find this for any M.
- What is the spectrum if we allow k, l to run over all integers?

Solution

The eigenvalue equation is

$$u_{k+1} + u_{k-1} = \lambda u_k$$

The solution is

$$u_k = A_1 \rho_1^k + A_2 \rho_2^k$$

where$\rho_{1,2}$ are solutions of the quadratic

$$\rho + \frac{1}{\rho} = \lambda$$

That is, $\rho_1 = \frac{1}{\rho_2}$.Imposing the various b.c. will determine the spectrum. If we use instead the continued fraction method the recursion relation

$$R_k(\lambda) = (\alpha_k - \lambda) - \frac{\beta_k \gamma_k}{R_{k-1}(\lambda)}$$

becomes

$$R_k(\lambda) = -\lambda - \frac{1}{R_{k-1}(\lambda)}$$

These rational functions tend to a limit $R(\lambda)$ that is no longer rational as $M \to \infty$:

$$R(\lambda) = -\lambda - \frac{1}{R(\lambda)}, \implies R(\lambda) = \frac{-\lambda \pm \sqrt{\lambda^2 - 4}}{2}$$

The branchcut shows that T has a continuous spectrum $[-2, 2]$.

2.3 An Infinite Dimensional Determinant

Find det $\left[\frac{d^2}{dx^2} + \lambda\right]$ on the interval $[0, \pi]$ subject to the mixed boundary condition $\psi'(0) = 0, \psi(\pi) = 0$. It is useful to know that $\zeta(s, a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}$ has the expansion $\zeta(s, \frac{1}{2}) \sim \left[-\frac{1}{2} \log 2\right] s + \mathrm{O}(s^2)$.

Solution

The solution to the ODE with the b.c. $\psi'(0) = 0, \psi(0) = 1$ is

$$\psi(x) = \cos\sqrt{\lambda}x$$

As expected, this is an entire function: there is actually no branch cut at $\lambda = 0$ because $\cos\sqrt{x} \approx 1 - \frac{\lambda}{2!}x^2 + \cdots$ there. The quantity that vanishes at the other boundary is

$$D(\lambda) = \cos\sqrt{\lambda}\pi$$

The product formula for cosine gives

$$D(\lambda) = \prod_{k=0}^{\infty}\left[1 - \frac{\lambda}{\left(k+\frac{1}{2}\right)^2}\right]$$

From the given expansion

$$\zeta'(0, \frac{1}{2}) = -\frac{1}{2}\log 2$$

$$e^{-2\zeta(0,\frac{1}{2})} = 2$$

$$\begin{aligned}
\det\left[\frac{d^2}{dx^2} + \lambda\right] &= \prod_{k=0}^{\infty}\left[-\left(k+\frac{1}{2}\right)^2 + \lambda\right] \\
&= \prod_{k=0}^{\infty}\left\{-\left(k+\frac{1}{2}\right)^2\right\} D(\lambda) \\
&= e^{i\pi\zeta(0) - 2\zeta'(0,\frac{1}{2})} D(\lambda)
\end{aligned}$$

The first term term in the exponential takes care of the signs in the product. Since

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0, \frac{1}{2}) = -\frac{1}{2}\log 2$$

$$\det\left[\frac{d^2}{dx^2} + \lambda\right] = -2i\cos\left[\sqrt{\lambda}\pi\right]$$

2.4 Airy Function

2.4.1 Power series

Find two linearly independent solutions to Airy's equation $\psi'' = x\psi$ as power series in x. Find the linear combination of these series that vanishes as $x \to \infty$. (You may have to devise an analogue of Laplace's method for infinite series.)

Solution

Step1: The series solution

The series

$$\psi(x) = \sum_{n=0}^{\infty} a_n x^n$$

gives the recursion relation

$$(n+1)(n+2)a_{n+2} = a_{n-1}, \quad n = 0, 1, 2 \cdots$$

It is understood here that $a_{-1} = 0$. It follows that $a_2 = 0 \implies a_{3k+2} = 0$.Setting $a_{3k} = b_k, a_{3k+1} = c_k$ we get

$$b_k = \frac{b_{k-1}}{(3k)(3k-1)}, \quad c_k = \frac{c_{k-1}}{(3k)(3k+1)}$$

Thus

$$b_k = \frac{1}{9}\frac{1}{k}\frac{1}{k-\frac{1}{3}}b_{k-1}$$

Recall that

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

s0 that

$$b_k = 9^{-k}\frac{1}{k!}\frac{1}{\Gamma\left(k+\frac{2}{3}\right)}b_0$$

and similarly

$$c_k = 9^{-k}\frac{1}{k!}\frac{1}{\Gamma\left(k+\frac{4}{3}\right)}$$

Thus

$$\psi(x) = b_0 \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k+\frac{2}{3}\right)}\frac{1}{k!}\left(\frac{x^3}{9}\right)^k + c_0 x \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k+\frac{4}{3}\right)}\frac{1}{k!}\left(\frac{x^3}{9}\right)^k$$

Step2: The asymptotic behavior of the series

For large x we must look at the large $k$behavior. We can use Stirling's formula to get

$$\log\left[\frac{1}{k!}\frac{1}{\Gamma(k+\frac{2}{3})}\right] \sim (2 - 2\log k)k + \frac{2}{3}\log\frac{1}{k} + O(k^0)$$

$$\log\left[\frac{1}{k!}\frac{1}{\Gamma(k+\frac{4}{3})}\right] \sim (2-2\log k)k + \frac{4}{3}\log\frac{1}{k} + O(k^0)$$

$$\psi(x) = \sum_{k=0}^{\infty} e^{k[3\log x - \log 9 + 2 - 2\log k]}\left\{b_0 k^{-\frac{2}{3}} + xc_0 k^{-\frac{4}{3}}\right\} + \cdots$$

Step 3: Approximation of the series by an integral

For large k we can approximate the sum by an integral.

$$\psi(x) \approx \int e^{S(k)} f(k) dk$$

$$S(k) = k\left[3\log x - \log 9 + 2 - 2\log k\right]$$

$$f(k) = b_0 k^{-\frac{2}{3}} + xc_0 k^{-\frac{4}{3}} S$$

Step4: Laplace's Method

The maximum of the exponent occurs at

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$$3\log x - \log 9 - 2\log k = 0$$

That is, at about

$$k(x) \sim \frac{x^{\frac{3}{2}}}{3}$$

In order for the leading contribution to vanish at large x

$$f(k(x)) = b_0\left[k(x)\right]^{-\frac{2}{3}} + xc_0[k(x)]^{-\frac{4}{3}} = 0$$

That is

$$b_0 + c_0 x[k(x)]^{-\frac{2}{3}} = 0$$

or

$$b_0 + c_0 3^{\frac{2}{3}} = 0.$$

This is the linear relation we seek.

$$\mathrm{Ai}(x) = b_0\left[\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(k+\frac{2}{3}\right)}\frac{1}{k!}\left(\frac{x^3}{9}\right)^k - 3^{-\frac{2}{3}}x\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(k+\frac{4}{3}\right)}\frac{1}{k!}\left(\frac{x^3}{9}\right)^k\right]$$

This predicts that

$$\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} = -\frac{3^{-\frac{2}{3}}\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})} \approx -0.729011$$

2.4.2 Fourier Integral

Expand the integrand in

$$\mathrm{Ai}(x) = \int_0^\infty \cos[kx + \frac{1}{3}k^3]\frac{dk}{\pi}$$

as a power series in x and evaluate the integral term by term. Compare with the answer in the previous part.

2.4.3 Quarkonium

The potential between heavy quarks (b or c) and anti-quarks is to a good approximation linear. Also, non-relativistic quantum mechanics applies to them. Find the energy levels for the radial excitations of zero orbital angular momentum for this system, in terms zeros of Ai' or Ai. Derive also a WKB approximation for these energies.

3 Problem set 3(Due March 28 2012)

3.1 Weyl Transform

Starting with the rule $e^{i[a\cdot q+ib\cdot p]} \to \hat{U}(a,b) = e^{\frac{1}{2}i\hbar a\cdot b}e^{ia\hat{q}}e^{ib\hat{p}}$ to turn functions into operators derive the formula for the integral kernel of the quantum operator corresponding to any function on phase

$$A(q,q') = \int \tilde{A}\left(\frac{q+q'}{2}, p\right) e^{\frac{i}{\hbar}p\cdot(q-q')}\frac{dp}{(2\pi\hbar)^n}$$

(i.e., fill in the details of the argument in the notes.) Prove its inverse

$$\tilde{A}(q,p) = \int A\left(q+\frac{u}{2}, q-\frac{u}{2}\right) e^{-\frac{i}{\hbar}p\cdot u}du$$

Show that the trace of an operator is related to its symbol by

$$\mathrm{tr}\hat{A} \equiv \int A(q,q)dq = \int \tilde{A}\left(q,p\right)\frac{dqdp}{(2\pi\hbar)^n}$$

3.2 Higher order terms in the star product

Given any pair of functions $f, g : R^{2n} \to C$ on phase space, and given a positive number n,write a symbolic program which will calculate the star product to order $\hbar^n$. Verify it on low order polynomials (quadratic or quartic).

3.3 Normal Ordered Star Product

It is possible to think of observables as functions of $z = \frac{q-ip}{\sqrt{2}}, \bar{z} = \frac{q+ip}{\sqrt{2}}$ and wavefunctions as analytic functions of z. The creation-annihilation operators are

$$a = \frac{\partial}{\partial z}, a^\dagger = z$$

One commonly used quantization rule is **normal ordering**

$$z^m \bar{z}^n \to a^{\dagger m} a^n, \quad m, n > 0$$

i.e., the annihilation operator acts first. Using this rule, derive a formula for the star product of two functions. Find the spectrum of the harmonic oscillator hamiltonian

$$\tilde{H} = z\bar{z}$$

this formalism.

3.4