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Diagonalization and Renormalization of the One-Boson Lee Model

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Renormalization in simple quantum mechanical systems is illustrated by diagonalizing the Hamiltonians for the displaced harmonic oscillator and the one-boson Lee model. The diagonalizing transformation for the one-boson Lee model is found by generalizing its dressing transformation. The renormalized fermion masses in the diagonalized Hamiltonian are found to be q numbers.

I. INTRODUCTION

The concept of renormalization is usually associated with the theory of interacting quantum fields and, thus, is usually discussed only in courses in advanced quantum mechanics. However, there exist physically interesting quantum mechanical systems which can be solved by students in a first course in quantum mechanics, and which also exhibit renormalization.

Two such models are discussed in this note. They are the field theoretic interpretation of the shifted harmonic oscillator,[1] and the one-boson Lee model,[2] which is also the model for optical spin-resonance in the rotating wave approximation.[3] The solutions of these models, i.e., the eigenfunctions and eigenvalues, have been previously obtained.[4] However, explicit diagonalization of the Hamiltonians immediately leads to mass renormalization in a transparent manner.

The canonical transformation which diagonalizes the Hamiltonian of the shifted harmonic oscillator is well known.[5] It is also the clothing or dressing transformation for this Hamiltonian.[6] The diagonalizing transformation for the one-boson Lee model Hamiltonian was not previously known. However, the simplicity of the model and the equality of the dressing and diagonalizing transformations for the harmonic oscillator led to the conjecture that the diagonalizing transformation would be simply related to the dressing transformation.

In Sec. II the displaced harmonic oscillator is analyzed in detail in order to introduce the operator techniques. The renormalized mass follows immediately upon diagonalization.

In Sec. III the dressing of the one-boson Lee model is carried out. The model's Hamiltonian is then diagonalized by choosing the dressing parameter to be a q number. This results in q number fermion masses. Similar behavior was found for the fermion mass in the demonstration of the equivalence of pseudovector and pseudoscalar meson coupling,[7] which is based on a canonical transformation similar to the diagonalizing one for the Lee model.

II. SHIFTED HARMONIC OSCILLATOR

The displaced harmonic oscillator can be cast into field-theoretic form by considering the coupling between a one-mode boson field and a one-mode fermion field. The Hamiltonian for this system is

$$H=M_0\psi^+\psi+\omega a^+a+g\psi^+\psi(a+a^+),$$

and the nonvanishing commutation relations are

$$\{\psi,\psi^+\}=1,\qquad [a,a^+]=1.$$

The diagonalizing and dressing transformations for this Hamiltonian are the same. They are generated by

$$V(\lambda)=\exp\lambda\psi^+\psi(a^+-a)\equiv\exp\lambda S,$$

and the dressed operators are defined by the transformation

$$X(\lambda)=V(\lambda)XV(\lambda)^{-1}.$$

These dressed operators are found by integrating the differential equations

$$(d/d\lambda)X(\lambda)=V(\lambda)[S,X]V(\lambda)^{-1}$$

subject to the boundary conditions $X(0)=X$.

For this particular case, the equations for the dressed operators are:

$$(d/d\lambda)\psi(\lambda)=-\psi(\lambda)[a(\lambda)^+-a(\lambda)]$$
$$=-\psi(\lambda)[a^+-a]$$

and

$$(d/d\lambda)\,a(\lambda) = -\psi(\lambda)^+\psi(\lambda) = -\psi^+\psi.$$

The solutions are:

$$\psi(\lambda) = \psi \exp[-\lambda(a^+ - a)]$$

and

$$a(\lambda) = a - \lambda\psi^+\psi.$$

These equations are now inverted so as to express the bare operators in terms of the dressed ones. Then, the Hamiltonian in terms of the dressed operators is

$$H = [M_0 + \omega\lambda^2 + 2g\lambda]\psi(\lambda)^+\psi(\lambda) + \omega a(\lambda)^+ a(\lambda) + [\lambda\omega + g][a(\lambda)^+ + a(\lambda)]\psi(\lambda)^+\psi(\lambda).$$

Clearly, the Hamiltonian is diagonal if the choice

$$\lambda = -g/\omega$$

is made. This choice will also result from the dressing condition, i.e.,

$$H[\psi(\lambda), a(\lambda)]\psi(\lambda)^+ | 0\rangle = M_R\psi(\lambda)^+ | 0\rangle.$$

The final form of the Hamiltonian is

$$H = M_R\psi(\lambda)^+\psi(\lambda) + \omega a(\lambda)^+ a(\lambda),$$

where

$$M_R = M_0 - g^2/\omega.$$

Thus, the fermion mass is renormalized by the attachment of a boson cloud which results from the interaction.

III. ONE-BOSON LEE MODEL

The one-boson Lee model is described by the Hamiltonian

$$H = M_v^0 V^+ V + M_N N^+ N + a^+ a + g[aV^+ N + a^+ N^+ V],$$

where the nonvanishing commutators are

$$\{V, V^+\} = 1, \qquad \{N, N^+\} = 1, \qquad [a, a^+] = 1.$$

By using the well-known isomorphism between bilinear products of fermion operators and the Pauli spin matrices,[8] this Hamiltonian can be rewritten as

$$H = M[1 - T_z] + \Delta M\sigma_z + \omega a^+ a + g[a\sigma_+ + a^+\sigma_-],$$

where

$$\sigma_+ = V^+ N = \sigma_-^+, \qquad \sigma_z = V^+ V - N^+ N,$$
$$T_z = 1 - V^+ V - N^+ N,$$

and the new coefficients are

$$M = (M_v^0 + M_N)/2, \qquad \Delta M = (M_v^0 - M_N)/2.$$

The optical spin-resonance Hamiltonian, in the rotating wave approximation, is obtained by setting M to zero.

As discussed in the introduction, the interaction induced mass renormalization in this system becomes transparent once the Hamiltonian is diagonalized. The successful diagonalization of the Hamiltonian for the displaced harmonic oscillator by its dressing transformation provides the clue for the diagonalization of the one-boson Lee model. The dressing transformation is known to be generated by[5]

$$V(\lambda) = \exp\lambda[aV^+ N - a^+ N^+ V] = \exp\lambda[a\sigma_+ - a^+\sigma_-] \equiv \exp\lambda S.$$

The equations for the dressed operators in this case are:

$$(d/d\lambda)\,a(\lambda) = \sigma_-(\lambda),$$
$$(d/d\lambda)\,\sigma_-(\lambda) = a(\lambda)\,\sigma_z(\lambda),$$

and

$$(d/d\lambda)\,\sigma_z(\lambda) = -2[a(\lambda)\sigma_+(\lambda) + a(\lambda)^+\sigma_-(\lambda)].$$

Although these equations appear to be highly nonlinear, they are easily reduced to linear form by introducing the constants of the dressing. This is analogous to the reduction of the equations of motion to linear form by introduction of the nonlinear constants of the motion.[2] The dressing constant needed is

$$C \equiv \sigma_z + 2a^+ a = \sigma_z(\lambda) + 2a(\lambda)^+ a(\lambda).$$

Thus, once $\sigma_z(\lambda)$ is known, the dressed form for all of the operator combinations appearing in the Hamiltonian is known.

In order to find $\sigma_z(\lambda)$, the above constant of the motion is inserted into the equation

$$(d^2/d\lambda^2)\,\sigma_z(\lambda) = -2[\{\sigma_+(\lambda), \sigma_-(\lambda)\} + \{a(\lambda)^+, a(\lambda)\}\sigma_z(\lambda)],$$

to rewrite it as

$$(d^2/d\lambda^2)\,\sigma_z(\lambda) = -2[\{\sigma_+(\lambda), \sigma_-(\lambda)\} - \sigma_z(\lambda)^2] - 2[C+1]\sigma_z(\lambda).$$

By virtue of the anticommutation relation

$$\sigma_z(\lambda)^2 = \{\sigma_+(\lambda), \sigma_-(\lambda)\},$$

the equation for $\sigma_z(\lambda)$ reduces to

$$(d^2/d\lambda^2)\sigma_z(\lambda)+\Omega^2\sigma_z(\lambda)=0,$$

where

$$\Omega^2=2[\sigma_z+\{a, a^+\}].$$

The equation may be integrated as a c-number one because the commutation rule

$$[\sigma_z(\lambda), a(\lambda)]=0$$

implies that

$$[\sigma_z(\lambda), \Omega^2]=0.$$

The solution is

$$\sigma_z(\lambda)=\sigma_z \cos\lambda\Omega-2[a\sigma_+ + a^+\sigma_-](\sin\lambda\Omega/\Omega),$$

and there is no ordering ambiguity because of the commutation rule

$$[\Omega^2, a\sigma_+ + a\sigma]=0.$$

The other operator combinations which appear in the Hamiltonian follow immediately and are:

$$a(\lambda)^+a(\lambda)=a^+a+\tfrac{1}{2}\sigma_z(1-\cos\lambda\Omega) + (a\sigma_+ + a^+\sigma_-)(\sin\lambda\Omega/\Omega)$$

and

$$a(\lambda)\sigma_+(\lambda)+a(\lambda)^+\sigma_-(\lambda)=(a\sigma_+ + a^+\sigma_-)\cos\lambda\Omega + \tfrac{1}{2}\sigma_z\Omega \sin\Omega.$$

As before, these equations are now inverted so as to express the bare operators in terms of the dressed ones. Then, the Hamiltonian in terms of the dressed operators is

$$\begin{aligned}H=&M[1-T_z(\lambda)]\\&+\Delta M[\sigma_z(\lambda)\cos\lambda\Omega+2\gamma(\sin\lambda\Omega/\Omega)]\\&+\omega[a(\lambda)^+a(\lambda)+\tfrac{1}{2}\sigma_z(\lambda)(1-\cos\lambda\Omega)-\gamma(\sin\lambda\Omega/\Omega)]\\&+g[\gamma\cos\lambda\Omega-\tfrac{1}{2}\sigma_z(\lambda)\Omega\sin\lambda\Omega],\end{aligned}$$

where

$$\gamma\equiv a(\lambda)\sigma_+(\lambda)+a(\lambda)^+\sigma_-(\lambda).$$

Unlike the displaced harmonic oscillator, the dressing and diagonalizing transformations for the one-boson Lee model are not identical. If $V(\lambda)$ is to generate the dressing transformation, the dressing condition

$$HV(\lambda)^+|0\rangle=M_v^RV(\lambda)^+|0\rangle$$

must be satisfied. This yields the value

$$\lambda=\tfrac{1}{2}\tan^{-1}[2g/(\omega-2\Delta M)]$$

and the renormalized mass

$$M_v^R=M+\tfrac{1}{2}\omega+[(\Delta M-\tfrac{1}{2}\omega)^2+g^2]^{1/2},$$

which agrees with the usual sector analysis of this model.[9] However, this value for λ does *not* diagonalize the Hamiltonian.

In order to diagonalize the Hamiltonian it is necessary to choose λ to be the q number

$$\lambda=(1/\Omega)\tan^{-1}[g\Omega/(\omega-2\Delta M)].$$

Of course, this choice raises certain questions regarding the validity of the differential equations for the dressed operators.

Although this choice of λ commutes with S, it is true that the differential equations really are invalid because λ does not commute with the boson operator a. However, λ does commute with the relevant operator combinations a^+a, σ_z, and $a\sigma_+ + a^+\sigma_-$. These, of course, are just the operator combinations solved for. Thus, the results can be made rigorous by going back and starting from the dressing equations for these operator combinations.

The resultant diagonalized Hamiltonian is

$$\begin{aligned}H=&\omega a(\lambda)^+a(\lambda)+\tfrac{1}{2}\Gamma\sigma_z(\lambda)+M[1-T_z(\lambda)]\\=&\omega a(\lambda)^+a(\lambda)+\tfrac{1}{2}(M+\Gamma)V(\lambda)^+V(\lambda)\\&+\tfrac{1}{2}(M-\Gamma)N(\lambda)^+N(\lambda),\end{aligned}$$

where

$$\Gamma=\omega+[(2\Delta M-\omega)^2+(g\Omega)^2]^{1/2}.$$

Thus, in general, the interaction renormalized the mass of both fermions by an amount dependent on the particular state of the system. This is in marked contrast to the simple constant mass renormalization found for the displaced harmonic oscillator. [Note that the previously obtained mass M_v^R is regained when H acts on the state $V(\lambda)^+|0\rangle$.]

IV. CONCLUSION

The simple models analyzed in this paper show that the concept of renormalization can be introduced early in the career of the physics major. They also illustrate the power of canonical transformations, a tool often overlooked in quantum mechanics courses. Finally, the analysis of the one-boson Lee model should encourage an element of boldness on the part of the student. Too often he expects rigorous proofs before taking an innovative step rather than trusting in his abilities to produce them afterward.

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[6] Clothing transformations were applied to Hamiltonians which describe interacting fields in order to cast them into a form suitable for the application of perturbation theory. The effect of such a transformation is the expression of the canonically conjugate bare field operators as combinations of the dressed ones, i.e., the creation and annihilation operators for the one-particle states of the *total* Hamiltonian. Upon expressing the Hamiltonian in terms of dressed particle operators, the self-interactions are replaced by the appropriate renormalized masses. Clothing transformations are not unique because they are determined by the one-particle states. In particular, a diagonalizing transformation meets the requirements for being a clothing one also.

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Uses of Desk-Top Computers in Introductory Laboratories

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This note describes three of many uses to which the desk-top computer has been put in the introductory physics laboratory. The three uses involve an ac circuit laboratory, an equipotential mapping experiment, and an N-slit diffraction experiment. Typical desk-top computer-produced theory plots are shown; the student often compares his experimental measurements to theory by plotting his points directly on the computer plot.

INTRODUCTION

Recent advances in computer technology have made possible the desk-top computer. Such a computer is about the size of a calculating machine, but has memory storage and far greater calculational speed. A desk-top computer can be used as a calculator for arithmetic and trigonometric calculations without programming, or it can be programmed to do any series of steps repetitively.[1] Programming is done by pushing the computer's keys. The keys are either numbers, functions, or computer instructions. For example, in addition to the usual zero-through-nine keyboard, there are keys which automatically give trigonometric, exponential, and logarithmic functions for any argument. There are also keys to compare numerical values and to store or recall numbers from memory.

Such a desk-top computer is easy to use, easy to learn to program, and fairly powerful. Although desk-top computers do not replace large, high-speed computer installations, a very large number of introductory physics problems can be done on these small, inexpensive machines. Many problems have not been done previously at the introductory level simply because of the tedium involved in carrying out calculations.

This note illustrates three of many uses to which a desk-top computer has been put in the elementary-physics laboratories at Dartmouth. Even though Dartmouth has a superior time-sharing computer installation that more than 80% of the undergraduates use, the desk-top computer still fills a need. Small, straight-forward computations need not now be done on the big computer.

The programs discussed here used the Hewlett–Packard 9100 A desk-top calculator/computer; the programs are available from the author. The desk-top system that was used included not only the 9100 A computer but also an X–Y plotter attached to it.