Lecture #1: Quantum electrodynamics

QED Lagrangian and Hamiltonian, equation of motion, scattering matrix, perturbation theory, Feynman rules, probability and effective cross section.

Let us recall certain facts.
Free electron, Dirac equation:

$$\left(i\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)\psi(x) = 0, \text{ or } (\not{p} - m)\psi(x) = 0, \text{ where } \not{p} = \gamma_\mu p_\mu, p_\mu = i\frac{\partial}{\partial x_\mu}$$

Solution for a free electron - flat wave*spin factor, corresponding to different helicity states ($\lambda/2$):

$$\psi(x) = u_\lambda(p)e^{-ipx}, \text{ where } u_\lambda(p) = \sqrt{\varepsilon + m}\begin{pmatrix} \chi_\lambda \\ \dfrac{\lambda\,|\,p\,|}{\varepsilon + m}\chi_\lambda \end{pmatrix},\ \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

More importantly (and very usefully!)

$$\sum_{\lambda=1,2} u_\lambda(\pm p)\overline{u}_\lambda(\pm p) = \not{p} \pm m$$

Dirac equation for interacting electron:

$$\not{p} \rightarrow \not{p} + e\not{A},\ = i\gamma_\mu \frac{\partial}{\partial x_\mu} + e\gamma_\mu A_\mu = \gamma_\mu\left(i\frac{\partial}{\partial x_\mu} + eA_\mu\right)$$

$$\left\{\gamma_\mu\left(i\frac{\partial}{\partial x_\mu} + eA_\mu(x)\right) - m\right\}\psi(x) = 0$$

Photon- quantum of EM field:

$$F_{\mu\nu} = -F_{\nu\mu},\ F_{ik} = \varepsilon_{ikl}H_l,\ F_{i4} = -iE_i$$

Or $F_{\mu\nu} = \dfrac{\partial A_\nu}{\partial x_\mu} - \dfrac{\partial A_\mu}{\partial x_\nu}$

In vacuum:

$$\frac{\partial}{\partial x_\nu}F_{\mu\nu}(x) = 0$$

Solution for free photon – flat wave*spin factor (s=1):

$$A_\mu = e_\mu(k)e^{-ikx}, \text{ where } e_\mu k = 0, e_\mu^{\ *}e_\nu = \delta_{\mu\nu}$$

Interacting photons == Maxwell's equations: (HW: show that this indeed is the case):

$$\frac{\partial}{\partial x_\nu}F_{\mu\nu}(x) = j_\mu(x)$$

$$j_\mu(x) = -e\overline{\psi}(x)\gamma_\mu\psi(x)$$

Lagrange equation for field ϕ:

$$\frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial \phi,_\mu} - \frac{\partial L}{\partial \phi} = 0, \text{where } \frac{\partial \phi}{\partial x_\mu} = \phi,_\mu$$

We need a Lagrangian that will give us both Dirac and Maxwell equations (HW1):

$$L = -\frac{1}{4}F_{\mu\upsilon}F^{\mu\upsilon} + \overline{\psi}\left\{\gamma_\mu(i\frac{\partial}{\partial x_\mu} + eA_\mu) - m\right\}\psi$$

To decouple free particle pieces and interaction pieces let us rewrite it as

$$L = -\frac{1}{4}F_{\mu\upsilon}F^{\mu\upsilon} + \overline{\psi}\left\{i\gamma_\mu\frac{\partial}{\partial x_\mu} - m\right\}\psi + eA_\mu\overline{\psi}\gamma_\mu\psi$$

Interaction Lagrangian:

$$L_I = eA_\mu\overline{\psi}\gamma_\mu\psi = -j_\mu A_\mu$$

Interaction Hamiltonian:

$$H_I = -\int L_I d^3x$$

Scattering matrix

$$\Phi(\infty) = S(\infty,-\infty)\Phi(-\infty)$$

Perturbation theory tells us:

$$i\frac{\partial}{\partial t}S(t,t_0) = H_I(t)S(t,t_0)$$

The solution in its exact form:

$$S = T\exp\left(-i\int_{-\infty}^{+\infty}H_I(t)dt\right) = T\exp(i\int L_I(x)d^4x)$$

Please note that in its exact form the S-matrix is always unitary: $\exp(iR)$, where R is real.
If the interaction is weak we can use the perturbative expansion of the exponent:

$$S^{(n)} = \frac{i^n}{n!}\int d^4x_1\int d^4x_2...\int d^4x_n T(L_I(x_1)L_I(x_2)...L_I(x_n))$$

Thus, knowing (effective) Lagrangian one is able to write the scattering matrix.
Or, closer to home

$$S^{(n)} = \frac{(-ie)^n}{n!}\int\overline{\psi}(x_1)\not{A}(x_1)\psi(x_1)\times\overline{\psi}(x_2)\not{A}(x_2)\psi(x_2)\times...\times\overline{\psi}(x_n)\not{A}(x_n)\psi(x_n)d^4x_1d^4x_2...d^4x_n$$

Matrix element for process $i \rightarrow f$ can be written as

$$M_{if} = \left\langle i\,|\,S^{(n)}\,|\,f\right\rangle$$

Each fermion's wave function can be written as a sum of creation of fermions with certain quantum numbers (a_s)and destruction of antifermions ($b^+{}_s$). Similarly, each antifermion's wave function can be written as a sum of creation of antifermions with certain quantum numbers (b_s) and destruction of fermions ($a^+{}_s$):

$$\psi(x) = \sum_s a_s \psi_s{}^{(+)}(x) + \sum_r b_r{}^+ \psi_r{}^{(-)}(x);$$

$$\overline{\psi}(x) = \sum_s a_s{}^+ \overline{\psi}_s{}^{(+)}(x) + \sum_r b_r \overline{\psi}_r{}^{(-)}(x)$$

$$\langle 0_s^+ | \psi(x) | 1_s^+ \rangle = \psi_s{}^{(+)}(x), \langle 1_s^+ | \overline{\psi}(x) | 0_s^+ \rangle = \overline{\psi}_s{}^{(+)}(x),$$

$$\langle 0_r^- | \overline{\psi}(x) | 1_r^- \rangle = \overline{\psi}_r{}^{(-)}(x), \langle 1_r^- | \psi(x) | 0_r^- \rangle = \psi_r{}^{(-)}(x)$$

Now, in the expression for S we need to move all creation operators to the right and destruction operators to the left. Mind that bosons (photons) commute, while fermions (electrons) anticommute, thus you get a negative sign every time two fermions change place. Normally it does not matter, because we are interested in the square of the matrix element. It does matter when we'll need to consider interference between two or more diagrams. Need to be careful with the relative sign of permutations.

Field operators in momentum representation:

Inner photon line (Fourier transformation of the Green's function) $D_{\mu\nu}(k) = \dfrac{-ig_{\mu\nu}}{(k^2 + i0)}$

Massive boson $D_{\mu\nu}(p) = \dfrac{-i(g_{\mu\nu} - p_\mu p_\nu / M^2)}{(p^2 - M^2 + i0)}$

Inner electron line $S_{\alpha\beta}(p) = \dfrac{i}{\not{p} - m + i0} = \dfrac{i(\not{p} + m)}{p^2 - m^2 + i0}$

Photon operators $\langle 0_\lambda | A_\mu(x) | 1_\lambda \rangle = e_\mu^{(\lambda)}$

$\langle 1_\lambda | A_\mu(x) | 0_\lambda \rangle = e_\mu^{(\lambda)*}$

Electron operators $\langle 0_\mu^{(+)} | \psi(x) | 1_\mu^{(+)} \rangle = u^\mu(p)$

$\langle 1_\mu^{(+)} | \overline{\psi}(x) | 0_\mu^{(+)} \rangle = \overline{u}^\mu(p)$

$\langle 0_\mu^{(-)} | \overline{\psi}(x) | 1_\mu^{(+)} \rangle = \overline{v}^\mu(-p)$

$\langle 1_\mu^{(-)} | \psi(x) | 0_\mu^{(-)} \rangle = v^\mu(-p)$

Photon-fermion vertex $ie\gamma^\mu$ - energy momentum conservation at each vertex.
But…
Internal lines do not have to be "on mass shell". $p^2 \neq m^2$!!!

Useful gamma-matrix trivia:

$\gamma^{0+} = \gamma^0, \qquad (\gamma^0) = I$

$\overline{\psi} = \psi^+ \gamma^0$

$\gamma^{\mu+} = \gamma^0 \gamma^\mu \gamma^0, \qquad \gamma^{\mu+}\gamma^0 = \gamma^0 \gamma^\mu$

$Tr1 = 4, \qquad Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

$Tr(odd\#\gamma) = 0, \qquad Tr(\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\delta) = 4(g^{\mu\nu}g^{\sigma\delta} - g^{\mu\sigma}g^{\nu\delta} + g^{\mu\delta}g^{\nu\sigma})$