Measuring The Mass of Uranus by Observing the Revolution of the Moons Using Kepler's Third Law

Ryan Rubenzahl
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Abstract

In this project we determined the mass of Uranus to be $7.650 \times 10^{28} \pm 2.678 \times 10^{27}\ grams$, which is close to the accepted value of $8.681 \times 10^{28} grams$ by about 3.8σ. We determined this by observing the motions of Uranus' satellites, Titania and Oberon. In doing so, we validated Kepler's laws of planetary motion and Newton's laws of gravity by showing that the square of the period is proportional to the cube of the semimajor axis.

Theory

By imaging Uranus and it's satellites with the telescope at the Mees Observatory, we can track the movement of the satellites around the planet Uranus. Kepler's third law tells us that the orbital motion of a body is characterized by

$$P^2 = \frac{4\pi^2}{GM} a^3$$

Where P is the period of the orbiting body, a is its semimajor axis, M is the mass of the object the body is orbiting, and G is Newton's gravitational constant. So we see that if we can determine the period and semimajor axis of Uranus' satellites, we can use this relationship to determine the mass of the planet itself. We will need images from several different days to see the motion of the satellites. If we use a coordinate grid to track where the satellites are on each observing day, we can calculate the angles between these apparent positions using the law of cosines by measuring the separation of the satellites in the x and y directions. We can use the angles between the positions of the satellites relative to the planet with the time interval between observations to deduce an angular velocity around the planet, and thus a period. If we take the largest separation from our measurements to be the semimajor axis, due to the fact that the orbital plane of the satellites is not directly along our line of sight, we will have the required measurements to test Kepler's third law and determine the mass of Uranus.

Procedure

At the Mees Observatory, we used the CCD camera on the 24 inch telescope to image Uranus and its satellites. We took images over the course of several hours on three separate nights: October 3rd, 12th, and 28th. These images are then imported to the program *DS9* to be analyzed. In *DS9* we are able to record x and y coordinates of the planet and the satellites using the program's built in pixel coordinate system.

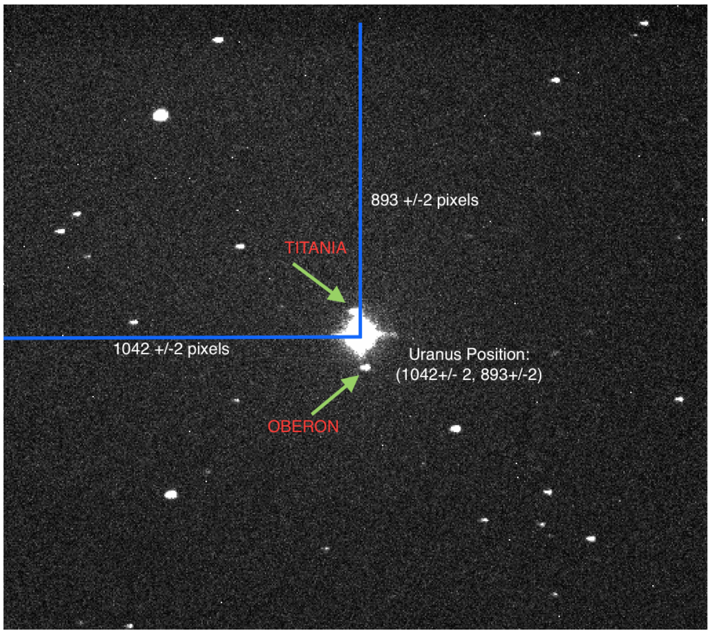


Figure 1: Image of Uranus, Titania, and Oberon on October 3rd, 2014 at 4:23:31 UTC. The position of the center of Uranus is 1042 ± 2 pixels from the left edge of the image and 893 ± 2 pixels from the top edge of the image. This image is inverted along the y axis from the actual orientation of Uranus in the sky.

To determine which moon is which in the image, I used a night sky simulator software called *Stellarium* to simulate the positions of the satellites at the times of our observations. This helped to determine that the images from October 3rd and 12th were inverted along the y-axis, and the images from October 28th were inverted along the x-axis and y-axis. Oberon and Titania were the only two satellites visible.

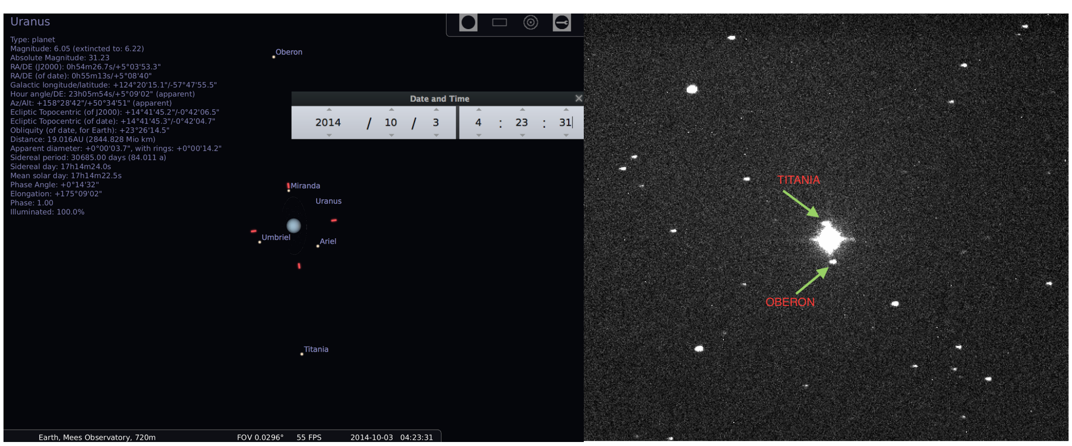


Figure 2: Comparison of the positions of the satellites from *Stellarium* and our Image. Miranda, Ariel, and Umbriel are lost in the glare of the planet. The CCD image is inverted vertically.

Data Analysis

Once we have the coordinates of Uranus, Oberon, and Titania for each image, we can calculate the separations in the x and y directions of each moon from the planet. We simply subtract the x coordinate of the planet from the x coordinate of the moon to find the x separation, and similarly for the y. Subtracting in this way will allow negative x separations to be to the left the planet and positive x separations to be to the right. Similarly, negative y separations will be below, and positive will be above. The uncertainty in the coordinates of objects is determined while measuring. For example, in this image, the square is the mouse cursor and is one pixel on each side. The display gives the coordinate of where the cursor is. To find the coordinate of the center of the object, I place the cursor at this point on the satellite and read the coordinate from the display. In this case, I would say the coordinate of the cursor is the center of the object ± 1 pixel, just from eyeballing where the center is. Although not perfect, this is a decent approximation of the center.

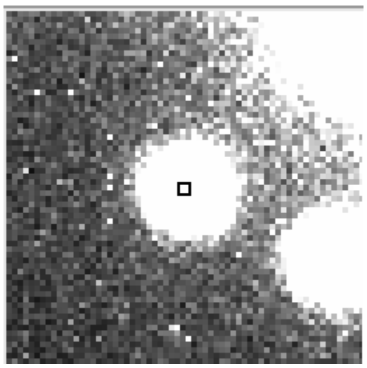


Figure 3: Finding the center of the object.

The coordinates of the objects are put into a spreadsheet to find the separations.

Table 1: Data from 10/3/2014 4:23:31 (Uranus_1225AM_Red.FIT)

Object	X (pixels)	Y (pixels)	σ_x (pixels)	σ_y (pixels)	S_x (pixels)	S_y (pixels)	$\sigma_{x\ separation}$ (pixels)	$\sigma_{y\ separation}$ (pixels)
Uranus	1042	893	2	2	N/A	N/A	N/A	N/A
Titania	1026	961	1	1	-16	68	2.236067975	2.236067977
Oberon	1058	801	2	1	16	-92	2.828427125	2.236067978

The errors in the separations are found using error propagation formula

$$\sigma_{x\ separation} = \sqrt{\frac{\partial S}{\partial x_U}^2 \sigma_{x_U}^{\ 2} + \frac{\partial S}{\partial x_m}^2 \sigma_{x_m}^{\ 2}}$$

Since the x separation is found using $S_x = X_m - X_U$, where X_m is the x coordinate of the moon and X_U is the x coordinate of Uranus, the partial derivative for both the moon term and the Uranus term are 1 and -1, and since they are squared the error formula reduces to

$$\sigma_{x\ separation} = \sqrt{\sigma_{x_U}^{\ 2} + \sigma_{x_m}^{\ 2}}$$

The error for the y separation is similar. Once the separation in the x and y direction and their errors have been calculated for all the images, a weighted mean for all of the images for each individual day can be calculated. The motion of the moons over the course of a couple hours will be negligible when comparing the positions over several weeks. By finding an average position for each night, we will be able to make our later calculations simpler. We have to use a weighted mean since each position that we measure has a different error assigned with it. To find the weighted mean for the first night of data, for example, we assign a weight to each data point

$$w_i = \frac{1}{\sigma_i^2},$$

and the weighted mean will be $\frac{\sum w_i y_i}{\sum w_i}$, and it's error is $\sigma = \frac{1}{\sqrt{\sum w_i}}$

So the calculations for the weighted mean of the separations for Titania are:

Table 2: Calculating weighted mean for 10/3/2014 data.

w_x	w_y	$S_x \cdot w_x$	$S_y \cdot w_y$
0.03448	0.03448	-0.41379	2.17241
0.03448	0.03448	-0.37931	2.24138
0.07692	0.07692	-1.15385	5.30769
0.12500	0.12500	-1.62500	8.75000
0.07692	0.07692	-0.92308	5.07692
0.20000	0.20000	-3.20000	13.60000
$\sum w_x$	$\sum w_y$	$\sum S_x \cdot w_x$	$\sum S_y \cdot w_y$
0.54781	0.54781	-7.69503	37.14841
$\overline{S_x}$ (pixels)	σ_x (pixels)	$\overline{S_y}$ (pixels)	σ_x (pixels)
-14.04685	1.35109	67.81237	1.35109

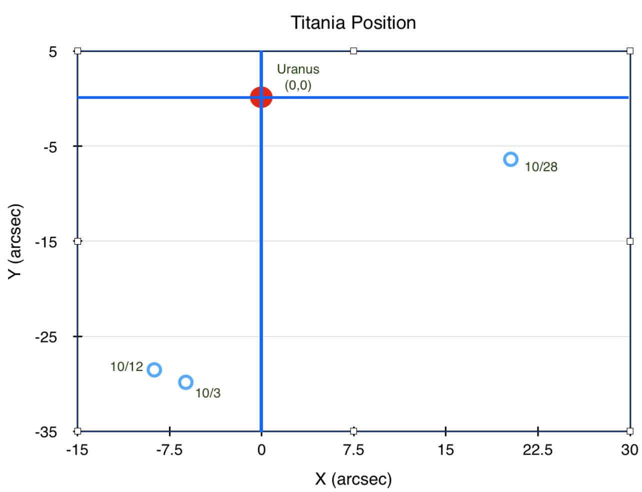
Since these separations are in pixels, we need to convert them to arcseconds for the rest of our calculations. The CCD on the telescope takes images in a 2048x2048 format, in which each pixel is equal to $0.44\ arcsec$, and a 4096x4096 format with $0.22\ arcsec/pixel$. The October 3rd and October 12th images are binned 2048x2048 while the October 28th images are binned 4096x4096. We convert the average separations for each day and each moon accordingly to get our final x and y separations.

We can also use these to calculate the overall separation using the pythagorean theorem, $\overline{S} = \sqrt{\overline{S_x}^2 + \overline{S_y}^2}$, and use error propagation for the error in this separation.

Table 3: x, y, and overall separations and their respective errors for each day and each moon.

	$\overline{S_x}$ (arsec)	σ_x (arsec)	$\overline{S_y}$ (arsec)	σ_y (arsec)	$\overline{S}$ (arsec)	σ_S (arsec)
DAY	Titania					
Oct 3	-6.18061	0.59448	-29.83744	0.59448	30.47085	0.59448
Oct 12	-8.74450	0.67341	-28.52865	0.67341	29.83874	0.67341
Oct 28	20.28757	0.55089	-6.38000	0.55089	21.26711	0.55089
DAY	Oberon					
Oct 3	6.30163	0.63989	40.75883	0.59448	41.24309	0.59558
Oct 12	-20.02991	0.67341	-22.93946	0.67341	30.45351	0.67341
Oct 28	-20.33514	0.55089	12.10000	0.55089	23.66279	0.55089

If we take the x and y separations as a new set of coordinates, then we can map the positions of the satellites over the course of the three observing days, with respect to Uranus which will be at the origin of the coordinate plane.



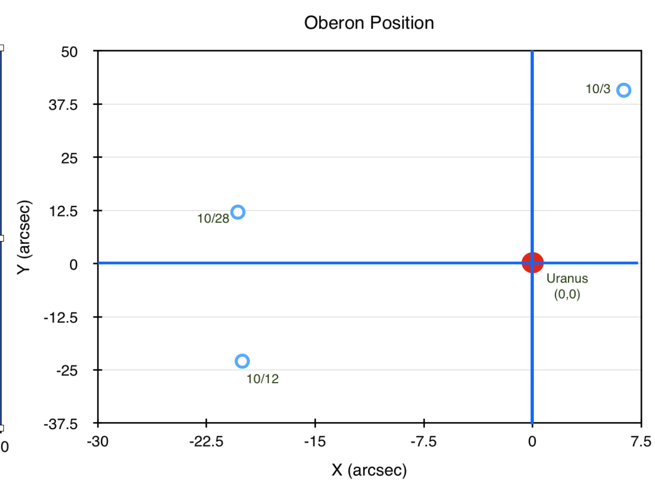


Figure 4: The positions of the moons Oberon and Titania on October 3rd, 12th, and 28th with respect to Uranus. This grid allows us to see the motion of the moons over a 9, 16, and 25 day time period.

Since we know each moon's distance from Uranus, if we calculate the separation between apparent positions, we will have a triangle with all three sides known. We could then use the law of cosines to determine the angle between the apparent positions from Uranus' perspective. We can calculate these distances easily, since we know the coordinates of both points, by using the distance formula.

$$D_{a\to b} = \sqrt{(x_1 - x_x)^2 + (y_1 - y_2)^2}$$

We will have to use error propagation to find the error in these distances, since there is an error for both the x and y coordinate.

$$\sigma_D = \sqrt{\frac{\partial D}{\partial x_1}^2 \sigma_{x_1}{}^2 + \frac{\partial D}{\partial x_2}^2 \sigma_{x_2}{}^2 + \frac{\partial D}{\partial y_1}^2 \sigma_{y_1}{}^2 + \frac{\partial D}{\partial y_2}^2 \sigma_{y_2}{}^2}$$

$$\text{where } \frac{\partial D}{\partial x_1}^2 = \frac{\partial D}{\partial x_2}^2 = [\tfrac{1}{2}(2(x_1 - x_2)\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}\ ]^2$$

$$\text{and } \frac{\partial D}{\partial y_1}^2 = \frac{\partial D}{\partial y_2}^2 = [\tfrac{1}{2}(2(y_1 - y_2)\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}]^2$$

Table 4: Distance between apparent positions

Titania			Oberon		
Day Interval	$D_{a\to b}$ (arcsec)	σ_D (arcsec)	Day Interval	$D_{a\to b}$ (arcsec)	σ_D (arcsec)
$3 \to 12$	2.87863	0.096685	$3 \to 12$	-	
$3 \to 28$	35.36688	0.895532	$3 \to 28$	39.12602	0.927266
$12 \to 28$	-		$12 \to 28$	35.04079	0.764482

We now have values for the lengths of all of the sides, so we can use the law of cosines to find the angle between apparent positions.

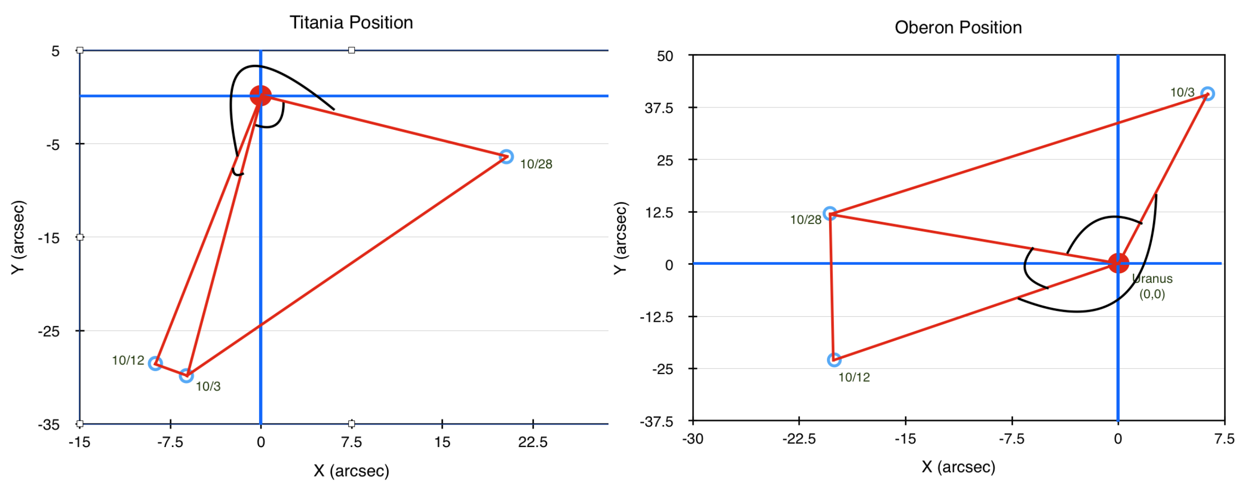


Figure 5: We have the lengths of all the sides, we now have to calculate the angles.

The law of cosine states that if a and b are the lengths of the separations and d is the distance between the apparent positions then the angle between the two separations is:

$$\theta_{a\to b} = cos^{-1}(-\ \frac{a^2+b^2-d^2}{2ab})$$

Since we have errors in a, b, and d, we have to use error propagation to find the error in $\theta_{a \to b}$. The error in the angle is given by:

$$\sigma_\theta = \sqrt{\frac{\partial\theta}{\partial a}^2 \sigma_a^2 + \frac{\partial\theta}{\partial b}^2 \sigma_b^2 + \frac{\partial\theta}{\partial d}^2 \sigma_d^2}$$

$$\text{where } \frac{\partial\theta}{\partial a} = \frac{\partial\theta}{\partial b} = \frac{-\frac{1}{b} + \frac{a^2+b^2-d^2}{2a^2 b}}{\sqrt{1 - \frac{(a^2+b^2-d^2)^2}{4a^2b^2}}}$$

$$\text{and } \frac{\partial\theta}{\partial d} = -\frac{d}{ab\sqrt{1 - \frac{(a^2+b^2-d^2)^2}{4a^2b^2}}}$$

From these equations, we can calculate the angles and their errors between these apparent positions. The third angle can be found by subtracting the other two from 360°:

Table 5: Angles between apparent positions and their errors.

Titania			Oberon		
θ between days	θ (degrees)	σ_θ (degrees)	Day Interval	θ (degrees)	σ_θ (degrees)
$3 \to 12$	5.33827	2.04858	$3 \to 12$	212.33751	3.94530
$3 \to 28$	84.24567	3.16951	$3 \to 28$	68.03490	2.63377
$12 \to 28$	270.41606	3.77392	$12 \to 28$	79.62759	2.93746

There may have been 0, 1, 2, 3, or more orbits that the moon could have made in between observations. So, using Titania as an example, Titania could have moved $5.34^\circ \pm 2.05^\circ$, $360^\circ + 5.34^\circ \pm 2.05^\circ$, $720^\circ + 5.34^\circ \pm 2.05^\circ$, $1080^\circ + 5.34^\circ \pm 2.05^\circ$, and so on from the 3rd to the 12th, an interval of 9 days. From the 12th to the 28th, 16 days, Titania could have moved $270.42 \pm 3.77^\circ$, $360^\circ + 270.42 \pm 3.77^\circ$, $720^\circ + 270.42 \pm 3.77^\circ$, etc, and from the 3rd to the 28th Titania could have moved the sum of these two: $275.75^\circ \pm 3.77^\circ$, $360^\circ + 275.75^\circ \pm 3.77^\circ$, $etc.$ What we need to do is make a table for the angular velocity of the satellite in it's orbit around the planet, for each possibility of in-between orbits. Only one combination of in-between orbits should give the same angular velocity for each time interval. This is a consequence of Kepler's second law and the conservation of angular momentum. The angular velocities won't be exactly equal as the orbit is not perfectly circular, and there are errors and approximations in our method, but the pattern should still be obvious. It is this time interval that is the one that describes the motion of the satellite around the planet, and the one that we will use to calculate the period of the satellite.

Table 6: The angular velocity, ω , from the angle between apparent positions of Titania, + 1 orbit, +2 orbits, etc. and the time interval: 9, 16, and 25 days. Values are in degrees per day.

Titania						
Time interval	T (days)	$\theta_{a \to b}$	$\theta_{a \to b}$ + 360	$\theta_{a \to b}$ + 720	$\theta_{a \to b}$ + 1080	$\theta_{a \to b}$ + 1440
3-12	9	0.59314	40.59314	80.59314	120.59314	160.59314
12-28	16	16.90100	39.40100	61.90100	84.40100	106.90100
3-28	25	11.03017	25.43017	39.83017	54.23017	68.63017

The pattern arises for one orbit between the 3rd and 12th, one orbit between the 12th and 28th, and two orbits between the 3rd and 28th (1 from 3-12 + 1 from 12-28). This combination tells us the orbital motion of the satellite. Each of these angular velocities have errors due to the error in the angles, so we can use σ_θ to find σ_ω through error propagation, the formula for which which reduces to:

$$\sigma_\omega = \frac{\sigma_\theta}{T}$$

So we have the angular velocities of Titania to be $40.59314 \pm 0.22762 \frac{deg}{day}$, $39.40100 \pm 0.23587 \frac{deg}{day}$, and $39.83017 \pm 0.17176 \frac{deg}{day}$. We can take a weighted mean to get an average angular velocity:

$$\overline{\omega_{Titania}} = 39.9287 \pm \ 0.1583 \frac{deg}{day}$$

Since this is the number of degrees Titania travels around Uranus each day, we can divide one orbit, 360° , by this angular velocity to get the number of days it takes to make one complete orbit, or the period of the orbit.

$$P_{Titania} = \frac{360^\circ}{39.9287 \pm 0.15826 \frac{deg}{day}} \Longrightarrow 9.0161 \pm 0.0357\ days$$

If we approximate the semimajor axis as the largest separation we measured for Titania, we have all of the variables we need to calculate the mass of Uranus. But first we have to convert arcseconds to radians, then to centimeters. We know there are 206265 arcseconds in one radian, so we can use $a = R\theta$ where θ is the separation in raidans, R is the distance from Earth to Uranus, and a is the semimajor axis.

$\theta = 30.4709 \pm 0.5945\ arcsec$, $R = 19.02AU$ (from *SolarSystemNOW*)

$$a_{Titania} = 19.02AU \cdot (30.4709 \pm 0.5945 arcsec) \Longrightarrow a = 4.203 \times 10^{10} \pm 8.201 \times 10^{8} cm$$

We can now use Kepler's third law to determine the mass of Uranus.

$$P^2 = \frac{4\pi^2}{GM}a^3 \Longrightarrow M = \frac{P^2}{a^3}\frac{G}{4\pi^2}$$

$$P_{Titania} = 9.0161 \pm 0.0357\ days \text{ and } a_{Titania} = 4.203 \times 10^{10} \pm 8.201 \times 10^{8} cm$$

Thus

$$M_{Uranus, from\ Titania} = 7.244 \times 10^{28} \pm 4.292 \times 10^{27} grams$$

If we calculate P and a for Oberon, using the same angular velocity method we used to find the period of Titania[1] and $a = R\theta$ with the largest measured θ, we find that

$$P_{Oberon} = 13.5878 \pm 0.0996\ days$$

and

$$a = 5.689 \times 10^{10} \pm 8.216 \times 10^{8} cm$$

and therefore

$$M_{Uranus, from\ Oberon} = 7.909 \times 10^{28} \pm 3.426 \times 10^{27} grams$$

We can take a weighted average of these two values to get our final result for the mass of Uranus:

$$M_{Uranus} = 7.650 \times 10^{28} \pm 2.678 \times 10^{27}\ grams$$

The accepted value for the mass of Uranus is $8.681 \times 10^{28} grams$, which is just outside 3σ from the value we calculated (our value $+3\sigma = 8.453 \times 10^{28} grams$, just shy of the actual value.) Our under approximation is most likely due to our approximation of the semimajor axis, as there easily could have been another point in the moon's orbit that we did not see in which it was further away. This would have raised the value for the mass to the point that our value would be statistically similar to the actual value.

[1] Tables for this calculation in Appendix

Conclusion

In conclusion, we came very close to determining the exact mass of Uranus by observing the motions of the moons. An interesting thing to note is that Titania nearly completed 3 orbits during our observation period, and Oberon nearly completed 2. If we look at the values we determined for the periods of the orbits, we find an interesting relationship.

$$\frac{P_{Oberon}}{P_{Titania}} = \frac{13.5878}{9.0161} = 1.5071 \approx \frac{3}{2}.$$

The two moons share a 3:2 mean-motion orbital resonance with one another; Titania orbits three times for every two orbits of Oberon. We can also see if our data confirms Kepler's third law of planetary motion. To validate Kepler's third law, we plot the square of the period of the two satellites against the cube of the semimajor axis. If Kepler is correct, then the result should be a linear relation with slope equal to $\frac{4\pi^2}{GM}$. For Uranus, this slope should be $6.818 \times 10^{-21} \frac{s^2}{cm^3}$.

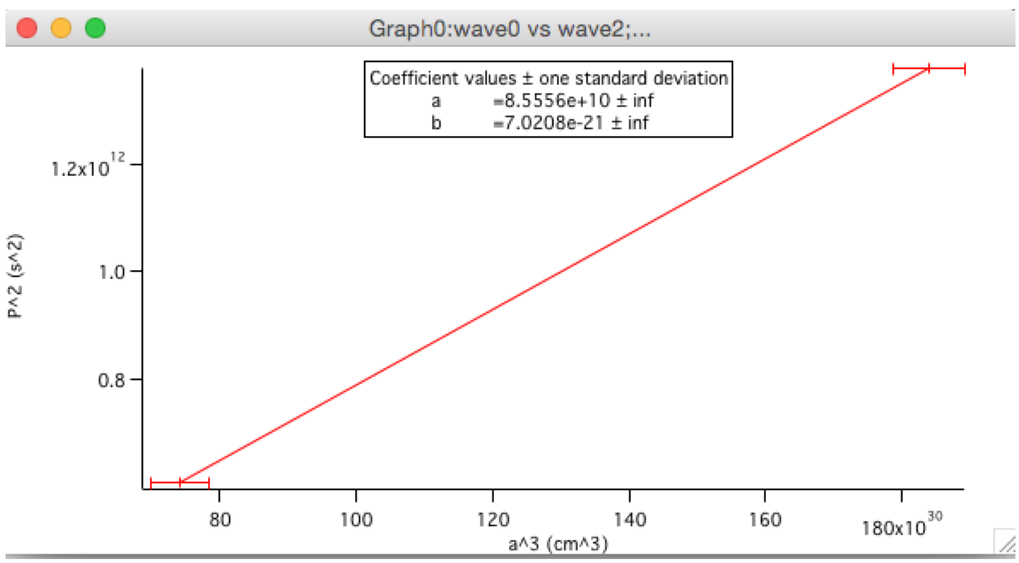


Figure 6: Plot of P^2 vs a^3, confirming Kepler's third law.

Plotting the data for Oberon and Titania, we see a clear linear relationship. Igor tells us that the slope of $7.021 \times 10^{-21} \frac{s^2}{cm^3}$. Our data thus confirms Kepler's third law, which tells us the slope for our data would be $6.818 \times 10^{-21} \frac{s^2}{cm^3}$. Kepler's law is a direct derivation of Newton's laws of gravity by balancing the gravitational force and the centripetal force of the object in orbit,

$$F = \frac{GMm}{r^2} = \frac{mv^2}{r} \rightarrow \frac{GM}{r} = v^2,\ v = \frac{d}{t} = \frac{2\pi r}{P} \Longrightarrow P^2 = \frac{4\pi^2}{GM}a^3 \quad \text{since} \ (r = a)$$

Our results support and validate Kepler's laws, and therefore also validate Newton's laws of gravity.

Appendix

Calculations for the angular velocity of Oberon (degrees per day):

Oberon						
Time interval	T (days)	$\theta_{a \to b}$	$\theta_{a \to b}$ + 360	$\theta_{a \to b}$ + 720	$\theta_{a \to b}$ + 1080	$\theta_{a \to b}$ + 1440
3-12	9	23.59306	63.59306	103.59306	143.59306	183.59306
12-28	16	4.97672	27.47672	49.97672	72.47672	94.97672
3-28	25	11.67860	26.07860	40.47860	54.87860	69.27860

The pattern arises when oberon only moves the angle between the positions on third and 12th, then orbits and moves the angle between the 12th and 28th, then moves the distance between the positions on 12th and 28th. So while Titania was almost able to complete three full orbits during our observation period, Oberon was only able to almost complete two full orbits. This is because the moons share a 3:2 mean-motion resonance.

Weighted mean of the angular velocity in degrees per day

Average Angular Velocity	error
26.49441	0.19427

Calculation of the orbital period of Oberon

$$\frac{360\ deg}{26.49441 \pm 0.19427 \frac{deg}{day}} \to 13.5878 \pm 0.0996\ days$$