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The reduction of thermal conductivity by magnetic fields in clusters of galaxies

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Summary. Tangled magnetic fields reduce heat fluxes in the intracluster medium by forcing heat to flow along the field lines rather than by more direct routes. The presence of field lines of different temperatures mixed together leads to an inhomogeneous multiphase description of the intracluster medium. A description of the magnetic field as a random walk leads to expressions for the mean temperature, temperature disperson and heat flux. If regions exist in the cluster which are isolated from the rest of the intracluster medium then these are natural sites for thermal instability leading to optical filamentation.

1 Introduction

Cooling flows are a widespread and important phenomenon in the universe, having implications for cosmology, galaxy formation and dark matter. Clusters of galaxies (Sarazin 1986) and elliptical galaxies (Thomas *et al.* 1986) contain vast quantities of hot gas that is in many cases accreting on to the centre of the system at rates up to several hundred solar masses per year. The fate of the accreted gas is uncertain: if it forms stars, this would explain the existence of the very large cD galaxies often found at the centres of cooling flow clusters. Our understanding of the cooling flow phenomenon is limited by an almost complete lack of knowledge of transport processes in the hot gas making up such systems. In this paper I concentrate on thermal conductivity in clusters of galaxies, although the discussion applies equally well to other transport processes and to elliptical galaxies and the interstellar medium.

Observations of the X-ray spectra of clusters of galaxies indicate that gas at a wide range of temperatures is present (see Sarazin 1986, section III.C). The thermal continuum is only well modelled if more than one temperature component is assumed to be present. Studies of X-ray line emission with the *Einstein* Solid-State Spectrometer (Mushotzky & Szymkowiak 1988) imply that cool gas is present that is not required to fit the continuum emission. In both ultraviolet (Fabian, Nulsen & Arnaud 1984) and optical (Lynds 1970; Heckman 1981) wavebands filaments of gas at $\sim 10^4$ K are seen. The presence of material coexisting at a wide variety of temperatures has led to a longstanding problem, as thermal conductivity at the Spitzer (1962) value is expected to erase temperature gradients and lead to an isothermal intracluster medium. Rephaeli & Wandel (1985) showed that clouds of gas at temperatures between 10^4 K and 10^7 K cannot coexist with hotter intracluster gas unless thermal conductivity is substantially reduced. Takahara & Takahara (1979) proposed an alternative to the cooling flow scenario in which cold gas in the centre of the system was being conductively evaporated by hotter outlying material. Binney & Cowie (1981) argued that thermal conductivity in M87 had to be reduced by nearly three orders of magnitude.

One would expect that thermal conductivity should play an important part in the energetics of the intracluster medium. The fact that it appears not to requires some explanation. There is no evidence that thermal conductivity differs significantly from the Spitzer (1962) value under a wide range of physical conditions, at least in directions parallel to the magnetic field. Classical conductivity appears to apply in the Interstellar Medium (Cowie & McKee 1977; McKee & Cowie 1977), in the outer atmospheres of stars (see, for example, Jordan *et al.* 1987) and in fusion devices such as Tokamaks (Haas & Thyagaraja 1986). In directions perpendicular to the magnetic field there appears to be anomalously enhanced transport (see Haas & Thyagaraja 1986 for a review). There is no evidence for a reduction in thermal conductivity in any of these cases. Despite this it is commonly assumed that tangled magnetic fields reduce thermal conductivity. In simple treatments (e.g. Pallister 1987) the procedure has been to multiply the Spitzer value of thermal conductivity by a global constant, assuming that the form of the Spitzer conductivity is correct and that the process that reduces thermal conductivity is independent of radius and other physical parameters, although Bregman & David (1988) assumed that the field lines would be stretched by the flow and lead to less reduction at the centre. Such models imply that reduction factors of 1 per cent are necessary. Although this conclusion has not gained universal acceptance (see Meiksin 1988) I will argue that there are good reasons for thinking that the macroscopic heat flux is indeed reduced by a large factor, whilst retaining the Spitzer (1962) conductivity at the microscopic level.

In Section 2 I consider the effect of a magnetic field on the flow of heat, and go on to use two simple models of the magnetic field to calculate the heat flux in a clean mathematical setting. The relevance of magnetic fields for thermal instability and the optical filament systems observed in clusters of galaxies is considered in Section 3. A short summary is given in Section 4.

2 The influence of magnetic fields

In the presence of a magnetic field, electrons in a plasma spiral around the field lines. They are free to move parallel to the field but excursions across the field are limited to one gyroradius r_g, which is usually very much smaller than the mean free path λ_e. For example, in the intracluster medium $r_g = 3\times 10^6\, T_8^{1/2} B_\mu^{-1}$ m whereas $\lambda_e = 7\times 10^{20}\, T_8^2 n_{-3}^{-1}$ m, where T_8 is the temperature in 10^8 K, n_{-3} is the electron density in 10^{-3} cm^{-3}, and B_μ is the magnetic field strength in μG, so that $r_g/\lambda_e \approx 10^{-14}$. Consequently thermal conductivity is almost completely eliminated in directions perpendicular to the magnetic field, and unaffected parallel to the field. I will therefore assume that the heat flux at all points is in the direction of the field, and proportional to the temperature gradient along the field, the proportionality constant being that given by Spitzer (1962). Since electrons can only move freely along field lines, the field lines are effectively insulated from each other and are independent. Hence I now analyse the thermal properties of a system of thermally isolated field lines.

2.1 THE RANDOM WALK APPROXIMATION

Consider a particularly simple model in which the magnetic field is approximated by a random walk. Two parallel plates of temperatures T_1 and T_2 are placed a distance $L = as$ apart, where s

is the step of the random walk. A constant conductivity is assumed. A field line is allowed to walk at random until it reaches one of the bounding plates. Three types of field lines will be found:

(i) Those which have both ends attached to the plate at temperature T_1. It is assumed that the temperature at all points along these field lines is T_1.

(ii) Those which have both ends attached to the plate at temperature T_2. It is assumed that the temperature at all points along these field lines is T_2.

(iii) Those which make it across from one plate to the other. The temperature distribution along such a field line is taken to be proportional to the distance along the field line.

When successive steps are uncorrelated, the problem as set up here is well known as the Gambler's ruin problem (Feller 1950). Two gamblers, one with capital k and the opponent with capital $a-k$ gamble for one unit of capital with equal probabilities of either player winning. The course of the game, described by the capital of the first player, is a bounded random walk with $k=0$ corresponding to ruin and $k=a$ to victory. The expected duration of the game can be found. For the thermal conductivity problem winning (ruin) corresponds to reaching the plate at $T_2(T_1)$ and the duration of the game to the path length between the plates.

The probability of reaching the plate at T_2 starting from position k is

$$P_k = k/a$$

and of reaching the other plate is

$$P_k = 1 - k/a.$$

Therefore at points k steps distant from the plate at T_1 a fraction $(1-k/a)^2$ of the field lines are at temperature T_1, a fraction $(k/a)^2$ are at T_2 and a fraction $2(k/a)(1-k/a)$ are at intermediate temperatures. The macroscopic temperature k steps from the first plate is taken to be the average temperature of all field lines through k. For large separations of the plate much of the gradient in the average temperature is given by the varying fractions of field lines of types (i) and (ii). This clearly demonstrates that a large-scale temperature gradient will not necessarily lead to a significant heat flux, even if the conductivity is large.

The probability of reaching plate 2 or 1 from position k in exactly m or n steps is written as P_{km} and Q_{kn} respectively. By assumption (iii) above the temperature at a point on a field line at k that is m steps from T_2 and n steps from T_1 is

$$T_{mn} = \frac{mT_1 + nT_2}{m+n}, \tag{1}$$

so the average temperature at k is

$$\langle T_k \rangle = (1-k/a)^2 T_1 + (k/a)^2 T_2 + 2 \sum_{n=k}^{\infty} \sum_{m=a-k}^{\infty} Q_{kn} P_{km} T_{mn} \tag{2}$$

$$= T_1 + (k/a)(T_2 - T_1) + \Delta T,$$

where

$$\Delta T = (T_1 - T_2) \sum_{n=k}^{\infty} \sum_{m=a-k}^{\infty} Q_{kn} P_{\mathrm{km}} \frac{m-n}{m+n}. \tag{3}$$

The average temperature distribution is approximately linear, with a small perturbation given by ΔT.

Near to any point there will be a large number of field lines which will have different temperatures. This leads to a very fine-scale multiphase picture of the intracluster medium (Nulsen 1986, 1988). This can be quantified by considering the temperature dispersion σ_T defined by

$$\sigma_T^2 = \langle (T - \langle T \rangle)^2 \rangle. \tag{4}$$

Fig. 1 shows graphs of $\langle T \rangle$ and σ_T as a function of distance between the plates. Details of how these quantities were obtained are given in Appendix B.

2.2 CALCULATION OF HEAT FLUX

In calculating the heat flux only those field lines that succeed in making it from one plate to the other need be considered. In particular, we may look at those field lines at $k=1$ that reach the plate at T_1 in one step, so that $n=1$. This picks up all the field lines as they leave plate 1. The

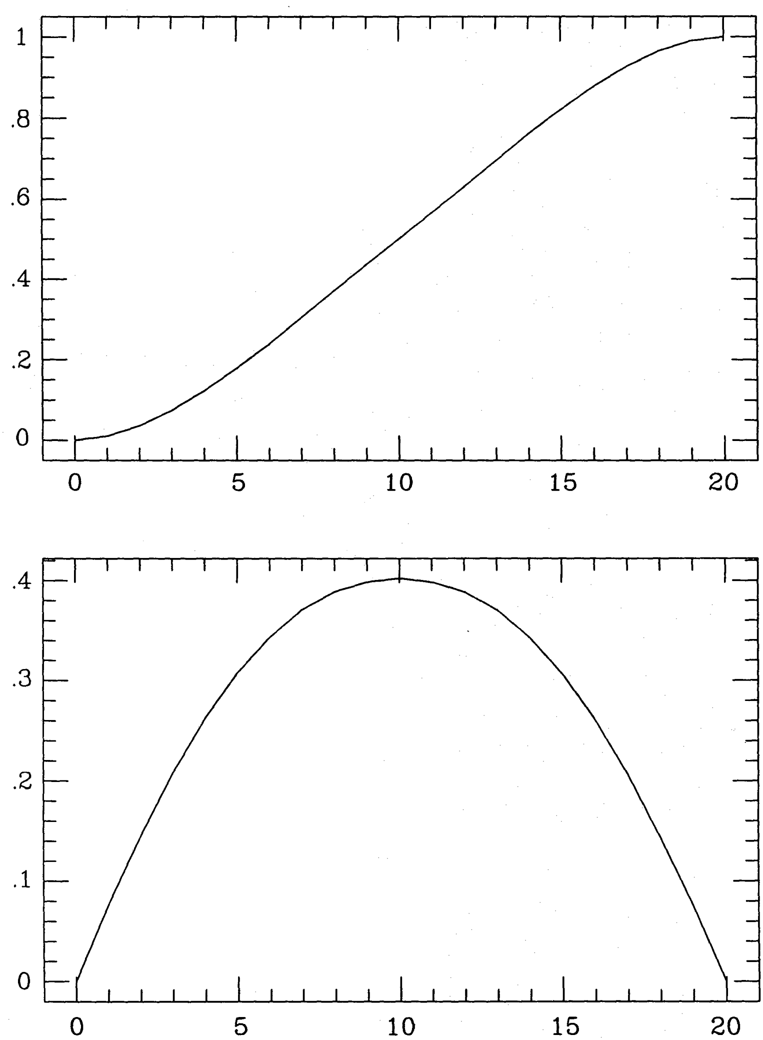

Figure 1. The temperature structure of a material between two parallel plates: the mean temperature as a function of distance between the plates (top); the temperature dispersion as a function of distance between the plates (bottom). Distances are in terms of the field step length and temperatures are in terms of the temperature difference between the plates.

heat flux between the plates is proportional to the probability of a field line succeeding in reaching plate 2 multiplied by the average temperature gradient along those that do. This is

$$P_1\nabla T = \sum_{m=a-1}^{\infty} \frac{P_{1m}}{m+1} \frac{(T_2 - T_1)}{s}. \tag{5}$$

To evaluate this expression we use the generating function $\mathbf{P}_k(x) = \Sigma_m P_{km} x^m$ where (Feller 1950)

$$\mathbf{P}_k(x) = \frac{\sin k\phi}{\sin a\phi}; \qquad x \equiv \frac{1}{\cos \phi}. \tag{6}$$

This expression allows us to turn the sum in equation (5) into an integral:

$$\sum_{m=a-1}^{\infty} \frac{P_{1m}}{m+1} = \int_0^1 \sum P_{1m} x^m \, dx \tag{7}$$

$$= \int_0^1 \mathbf{P}_1(x) \, dx \tag{8}$$

$$= \int_0^\infty \frac{\tanh^2 y}{\sinh ay} \, dy,$$

where $\phi = iy$. Let $z = ay$. Then

$$\sum_{m=a-1}^{\infty} \frac{P_{1m}}{m+1} = \frac{1}{a} \int_0^\infty \frac{\tanh^2 (z/a)}{\sinh z} \, dz. \tag{9}$$

For large a we expand $\tanh^2(z/a) \sim z^2/a^2$ which will be a good approximation provided that the denominator kills the integrand before the expansion breaks down. The integral is given in Gradshteyn & Ryzhik (1965; 3.523.1) so the sum is approximately

$$\sum_{m=a-1}^{\infty} \frac{P_{1m}}{m+1} \simeq \frac{4}{a^3}. \tag{10}$$

The separation of the plates is $L = as$, so the total heat flux is

$$F = -4\kappa \left(\frac{s}{L}\right)^2 \frac{T_2 - T_1}{L}. \tag{11}$$

This result can be easily understood. The probability of a field line which leaves one plate reaching the other is $1/a$ and the average number of steps taken is a^2.

For an n-dimensional random walk between two parallel plates there is very little change in the mean temperature and temperature dispersion. The extra dimensions are treated by allowing the field lines to walk parallel to the plates, which is a draw in the terminology of the gambler's ruin problem. The heat flux calculation is changed only by the replacement

$$x \rightarrow x' \equiv \frac{n}{(n-1) + \cos \phi} \tag{12}$$

(see Cox & Miller 1965) which leads to the flux being reduced by a factor n.

The implications of this result are quite serious. First, there is no longer a thermal conductivity in the conventional sense: the heat flux is no longer simply inversely proportional to the separation of the plates, so that measuring the 'thermal conductivity' of a sample of a material will lead to a result dependent on the size of the sample. Secondly, if the step length is small compared to the separation of the plates then there is a substantial reduction in the heat flux compared to that expected by taking the macroscopic temperature gradient and applying classical conductivity.

2.3 THE CORRELATED RANDOM WALK APPROXIMATION

Although the results of the simple random walk model give a good feel for the reduction process, the description of the magnetic field as a simple random walk is obviously unrealistic. As the next step I therefore describe the field as a correlated random walk, although still in one dimension. By a correlated random walk I mean a random walk in which the probability of going forward or back on step n depends on the direction chosen on the preceding step, so denoting steps forward and backward by + and − respectively, and the conditional probability of A and B on the preceding step as $P(A|B)$:

$$\begin{aligned} &P(+|+)=\alpha \qquad P(+|-)=\beta \\ &P(-|+)=\beta \qquad P(-|-)=\alpha. \end{aligned} \tag{13}$$

The relevant properties of a correlated random walk are summarized in Appendix A. The probability of a field line reaching the other plate is increased over an uncorrelated walk. Denoting the nature of the previous step by a superscript arrow, we again wish to pick up all the field lines as they leave the lower plate. This is done by considering only the generating function $\mathbf{P}_1^{\uparrow}(x)$, where from Appendix A

$$\mathbf{P}_1^{\uparrow}(x)=\frac{\alpha}{\beta}\frac{\sin\phi}{\sin\{\sin^{-1}[(\alpha/\beta)\sin\phi]+(a-1)\phi\}}. \tag{14}$$

Here

$$\cos\phi\equiv\frac{1+(\alpha-\beta)x^2}{2\alpha x}. \tag{15}$$

The heat flux is determined from equation (5) as before. The result is

$$F=-\frac{4\kappa}{a^3}\left(\frac{\alpha}{\beta}\right)^2\frac{T_2-T_1}{s}. \tag{16}$$

The heat flux is increased over the uncorrelated case by the factor $(\alpha/\beta)^2$. In the limit as the step length s goes to zero and the correlation goes to unity to keep a finite correlation length L_c (see Appendix A) the heat flux becomes

$$F=-16\kappa\left(\frac{L_c}{L}\right)^2\frac{(T_2-T_1)}{L}. \tag{17}$$

Thus in the more complicated and more realistic case of a correlated random walk the same scaling of the heat flux with the field's correlation length and the separation of the plates is obtained as for the simple random walk. It seems reasonable to extend this behaviour to realistic magnetic fields which are not amenable to so simple an analysis.

2.4 RANDOM WALKS IN SPHERICAL SYSTEMS

In spherical geometry the bounding plates are concentric spheres. If their radii are similar then the parallel plate analysis applies, if different then the variation of step length with radius becomes important. The strength and scale lengths of magnetic fields are expected to vary with radius in a cluster (Jaffe 1980). If simply transported by a cooling flow the field strength will increase and the scale length decrease as radius decreases (Soker & Sarazin 1988). If step length is proportional to r then the grid can be unfolded and the problem is equivalent to the parallel plate case with a position dependent step length. I therefore present results, in Fig. 2, from a model in which the scale length scales with radius.

In the unlikely case that the step length were independent of radius then the problem would be qualitatively different: it is well known (Feller 1950) that for a three-dimensional random

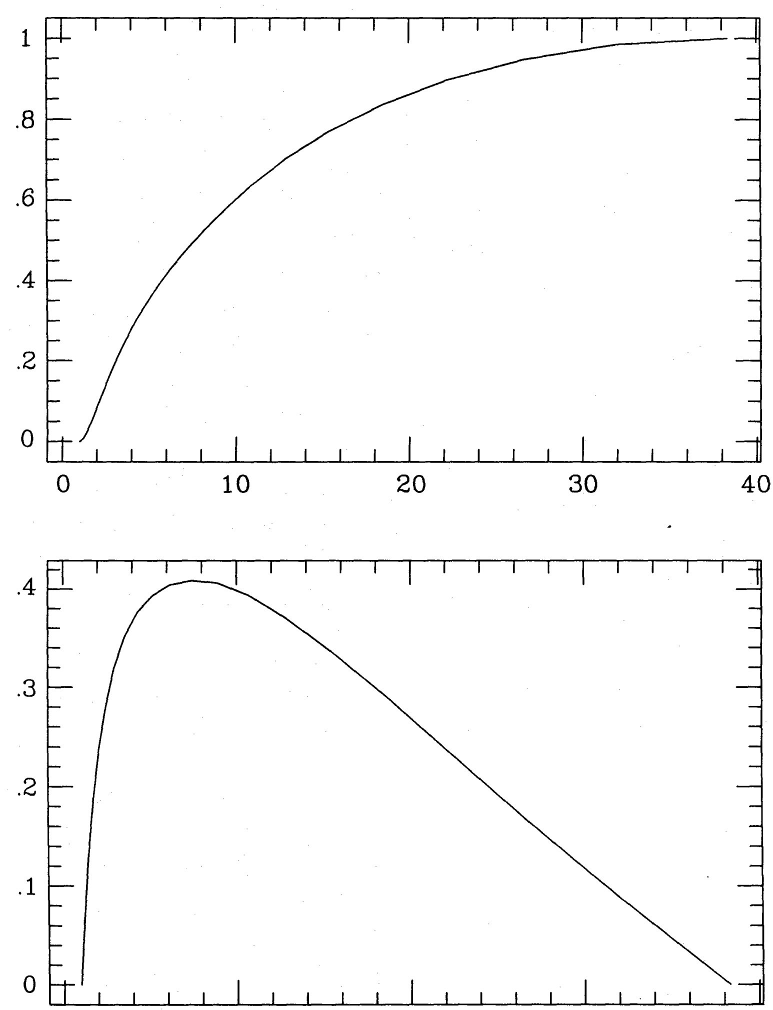

Figure 2. The temperature structure of a material between two concentric spheres. The mean temperature as a function of radius (top). The temperature dispersion as a function of radius (bottom). The step length is 0.2 times the radius at any point. Temperatures are in terms of the temperature difference between the plates.

walk on a cartesian grid approximately 1/3 of all walks leaving a point return and the remaining 2/3 never return, wandering to infinity. If the inner sphere is small compared to a step length approximately 2/3 of all field lines leaving it to make it to the outer sphere. The extra path length due to the walk will still act to reduce the heat flux.

The bounding plates are an abstraction necessary to make the problem tractable. I assume that the plates are equivalent to regions where the gas is mostly cold (at the centre, due to cooling) and mostly hot (at large radii, due to all thermal time-scales being longer than the age of the gas).

We may now estimate the reduction in heat flux. Consider a cluster in which the field scale length is proportional to radius, approximately 10–20 kpc at a few hundred kpc (Lawler & Dennison 1982; Dreher, Carilli & Perley 1987). Therefore the ratio of field scale size to system scale size is $\sim 1/10$ leading to the heat flux being reduced to 1 per cent of that derived from the macroscopic temperature gradient and Spitzer conductivity.

2.5 MULTIPHASE MODELS OF THE INTRACLUSTER MEDIUM

A multiphase medium was shown in Section 2.1 to be an inevitable consequence of the presence of tangled magnetic fields in an atmosphere with a temperature gradient. This has also been suggested from observational considerations. First, fitting the X-ray profile to models leads to a mass flow rate that varies with radius (Nulsen 1986, 1988; Thomas, Fabian & Nulsen 1987). It is argued that the flow is in a steady state and that this implies that mass is dropping out of the flow at a range of radii, and therefore that a range of gas properties exist at all radii. Secondly, the volume filling factor of cooling gas required to produce the observed Fe XVII line emission in Perseus is such that it would lead to the continuum emission being too centrally concentrated if the cool gas were the only phase present (Canizares, Markert & Donahue 1988).

For any model it is possible to calculate the standard deviation of the temperature from the mean. For the simple random walk this result is shown in Fig. 1, and for a more realistic case in spherical symmetry the results of Fig. 2 are obtained. These models clearly show that we expect to see a continuous range of temperatures at any radius, with the temperature dispersion being a substantial fraction of the temperature range supported by the system.

Due to the high efficiency of conductivity, the temperature along a field line will be constrained to be nearly homogeneous: small deviations from temperature equilibrium will be damped quickly. Perpendicular to the field there is no such constraint. If the gas is in pressure equilibrium then, with the temperature fixed, density inhomogeneities will be present in the flow. The dynamics of such inhomogeneities must now be taken into account. This will not be considered here but has been considered by Nulsen (1986) and Thomas (1988).

2.6 STOCHASTIC MAGNETIC FIELDS

A potentially serious problem arises with the degree of isolation of field lines from one another. The magnetic field is expected to be stochastic – initially neighbouring field lines follow divergent paths and soon become well separated. This leads to intermingling of field lines of different temperatures (on small scales) and allows electrons to diffuse across the mean field, even in the absence of collisions. Stochastic diffusion of the field lines must be taken into account in a complete discussion of the effect of a magnetic field on thermal conductivity. For an application to cosmic ray diffusion see Jokipii & Parker (1969) and for diffusion across mean fields in fusion devices see Rechester & Rosenbluth (1978).

Magnetic field stochasticity leads to transfer of electrons across the field, reducing the isolation of field lines from one another and blurring the classification of field lines of Section 2.1. As long as the conductivity perpendicular to the field is much less than the parallel conductivity the field lines will still be effectively isolated. The important quantity is the length scale D_{st} on which the stochastic divergence takes place – if this is short then the electrons will be well mixed, if long then the field lines retain their identity and the random walk description applies.

In clusters of galaxies we might expect a mean component not to be present, so that the entire field is stochastic. In this case the stochastic diffusion length is equal to the scale length of the field, $D_{st} \sim L_c$, and typically $\lambda_e \sim D_{st} \sim L_c$. Particles follow the stochastic diverging field lines until a collision occurs, and then are randomized and start off again on a different divergent path. Because $\lambda_e \sim D_{st}$ the particles only travel one divergence length between collisions, and in this distance two particles on originally neighbouring field lines will have only diverged by a few r_g. Relative to a mythical reference field line a particle will follow a random walk with step length a few r_g. A typical field line will be of length $N \sim 10^4$ steps so that the random walk perpendicular to the field will cause a separation of only $\sqrt{N} \sim 10^2$ steps, or only a few hundred r_g. Notice that the particles diffuse much more slowly across the mean field than do the field lines. With N being moderate the transverse thermal conductivity is still very much less than the conductivity parallel to the field, although much enhanced over the classical value. I conclude that magnetic field stochasticity is not important in the present context.

3 Instabilities and filamentation

It has been argued (Mathews & Bregman 1978; and others) that the intracluster medium is thermally unstable: a parcel of gas that is denser than its surroundings will cool faster. As it cools its density must increase to stay in pressure equilibrium, so that it cools even faster and a runaway process is established. It is further argued that these instabilities are the cause of the spectacular optical filament systems in cooling flow clusters and used as an extra argument in favour of the cooling flow picture.

The description of the evolution of small perturbations (Malagoli, Rosner & Bodo 1987; Balbus 1988; Tribble 1988) needs revision in the light of the description of the intracluster medium developed here. These authors consider linear perturbations around a smooth (i.e. homogeneous on small scales, although stratified) background equilibrium state. This is obviously inappropriate in the present case where the background state is highly inhomogeneous.

Thomas (1988) and Nulsen (1986; 1988) have considered the evolution of inhomogeneous flows, but these analyses did not consider the magnetic field in the context of the present paper. We can consider some aspects of the revised stability problem. Due to density variations some gas will cool more rapidly than its surroundings but thermal conductivity is effective along the field lines and will exert a stabilising influence. There is a critical wavelength below which perturbations are conductively stabilized which is approximately (Pallister 1987)

$$\lambda_{crit} \sim f^{1/2} T_7^{3/2} n_{-3}^{-1} \text{ Mpc}. \qquad (18)$$

Here f is a constant customarily introduced to allow for a reduction in thermal conductivity and $T_7 = T/10^7$ K.

Along field lines $f=1$ so only large-scale perturbations are not conductively damped. The scale length is along the field line, so that the gas in the centre of a cluster can cool because the total length along a tangled field line to hot gas can be longer than λ_{crit}.

Three perturbation scales are important. The first is the scale length of the field, as field lines walk between areas of different physical conditions on this scale. The second is the size of the cluster because this determines the scale of the background, and the third is the critical wavelength defined in equation (18). If thermal conductivity is fully effective along field lines then small-scale perturbations are stabilized. Occasionally, however, magnetic fields have conspired to so reduce the heat flux that the perturbation can grow. Only if the magnetic field sufficiently reduces the heat flux into an element of gas can that element of gas be unstable. The magnetic field has a twofold effect, determining the spectrum of density inhomogeneities (these should not be regarded as perturbations – they are a necessary part of an atmosphere threaded by a tangled magnetic field) and the local stability of the gas. It does these two tasks independently: there need be no correlation between density and stability.

We are therefore led to the hypothesis that the instabilities leading to the observed filaments are suppressed except at specific places where the magnetic field geometry especially restricts the flow of heat. These will be locations where, although thermal conductivity may be fully effective, the gas is not well connected to the rest of the intracluster medium. Specific field geometries ('islands' or tori) are involved. Alternatively, local suppression of thermal conductivity (e.g. by a magnetic mirror) restricts the flow of heat.

This hypothesis has several attractions. It naturally explains the observed sizes of the filaments, by relating them to the magnetic field structure. Filament sizes should be similar to the magnetic field scale length. Magnetic fields have scale sizes of the order of 10 kpc (Lawler & Dennison 1982; Dreher *et al.* 1987) which is indeed of the order of the filament size. This model does not require any deviation from the known properties of thermal conductivity in the intracluster medium, and introduces no free parameters. The major disadvantage is that we know very little about the magnetic field structure, and cannot at present predict any properties of the filaments using the theory.

4 Conclusions

Cooling flows contain material at a wide range of temperatures. This is surprising in view of the ability of thermal conductivity at the classical (Spitzer 1962) value to erase all such temperature inhomogeneities on time-scales much shorter than a Hubble time. In order to avoid this problem, it has been common practice to assume that tangled magnetic fields reduce thermal conductivity, due to the almost complete suppression of thermal conductivity in directions perpendicular to the field.

In this paper I have considered the effect of a magnetic field on the flow of heat through the intracluster medium. The heat flux is reduced from the value that would be expected from the macroscopic temperature gradient because the heat is forced to travel along circuitous paths. The macroscopic temperature gradient is caused by differing amounts of hot and cold field lines at any radius. A multiphase intracluster medium is an inevitable consequence of a tangled magnetic field being present in a cluster. Models of the intracluster medium should take into account (i) the large-scale reduction in heat flux, (ii) the inhomogeneities caused by the magnetic field, and (iii) the fact that thermal conductivity is effective on small scales.

The model suffers from three deficiencies: (i) the imposition of the two bounding plates, (ii) the inaccurate description of the magnetic field, and (iii) the lack of dynamical considerations. These do not detract from the qualitative description of the process whereby thermal transport is reduced in clusters of galaxies. A more quantitative explanation would need a much better understanding of the precise state of magnetic fields within clusters of galaxies than we have at present.

The reduction mechanism is restricted to scales much larger than the scale length of the field. Observed optical filament systems have scale sizes very similar to the magnetic field. It was argued that a tangled magnetic field would contain regions where the local field geometry is such that a degree of isolation from the cluster at large is obtained. One expects thermal instability to develop in such disconnected regions. This naturally explains the observed filament sizes without any necessity for fine tuning the thermal conductivity suppression factor.

Further work needs to be done on the dynamical aspects of this problem, and on the detailed evolution of unstable material.

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Appendix A: The correlated random walk

Consider the situation in which the probability of making a step in a particular direction depends on the direction of the previous step. This can be expressed either by the conditional probabilities

$$P(+\mid+)=\alpha \qquad P(+\mid-)=\beta$$
$$P(-\mid+)=\beta \qquad P(-\mid-)=\alpha \tag{13}$$

or by the correlation coefficient between two successive steps

$$\langle\Delta_i\Delta_{i+1}\rangle=c. \tag{A1}$$

Although only successive steps are correlated this induces a correlation of c^n between displacements n steps apart. These two ways of expressing the correlation are related by

$$\alpha=\frac{1+c}{2}; \qquad \beta=\frac{1-c}{2}. \tag{A2}$$

We may consider the limit as $c\rightarrow 1$ and the step length s vanishes. Then

$$c^n\rightarrow e^{-L/L_c}$$

where

$$L=ns \quad \text{(constant)}; \qquad L_c=\frac{s}{1-c}.$$

The probability of winning (losing) starting from point k when the preceding step was up ($\uparrow$) is written as $P_k^{\uparrow}(Q_k^{\uparrow})$. Then the following recurrence relations hold

$$P_k^{\uparrow}=\alpha P_{k+1}^{\uparrow}+\beta P_{k-1}^{\downarrow}$$
$$P_k^{\downarrow}=\beta P_{k+1}^{\uparrow}+\alpha P_{k-1}^{\downarrow}$$
$$Q_k^{\uparrow}=\alpha Q_{k+1}^{\uparrow}+\beta Q_{k-1}^{\downarrow}$$
$$Q_k^{\downarrow}=\beta Q_{k+1}^{\uparrow}+\alpha Q_{k-1}^{\downarrow}. \tag{A3}$$

Symmetry dictates that $P_k^{\uparrow}$ is equal to $Q_{a-k}^{\downarrow}$, so that only P_k need be calculated. The boundary conditions on P_k are

$$P_0^{\downarrow}=0; \qquad P_a^{\uparrow}=1. \tag{A4}$$

There is no meaning to the expressions $P_0^{\uparrow}$ and $P_a^{\downarrow}$ since these involve points outside the sample space. It is easily verified that the solution to these equations is

$$P_k^{\uparrow}=\frac{k\beta+(\alpha-\beta)}{a\beta+(\alpha-\beta)}$$
$$P_k^{\downarrow}=\frac{k\beta}{a\beta+(\alpha-\beta)}, \tag{A5}$$

which reduces to the simple random walk result when α and β are identical.

We are interested in calculating the probabilities of winning at the nth step. Recurrence relations can again be given:

$$P^{\uparrow}_{k,n}=\alpha P^{\uparrow}_{k+1,n-1}+\beta P^{\downarrow}_{k-1,n-1}$$
$$P^{\downarrow}_{k,n}=\alpha P^{\uparrow}_{k+1,n-1}+\alpha P^{\downarrow}_{k-1,n-1}. \tag{A6}$$

Generating functions $\mathbf{P}$ are defined by

$$\mathbf{P}^{\uparrow}_{k}(x)=\sum_{n} P^{\uparrow}_{k,n}x^{n} \tag{A7}$$

and similarly for $\mathbf{P}^{\downarrow}_{k}$. Multiplying the recurrence relations (A6) by x^n and summing over all n leads to recurrence relations for the generating functions $\mathbf{P}$

$$\mathbf{P}^{\uparrow}_{k}(x)=x[\alpha\mathbf{P}^{\uparrow}_{k+1}(x)+\beta\mathbf{P}^{\downarrow}_{k-1}(x)]$$
$$\mathbf{P}^{\downarrow}_{k}(x)=x[\beta\mathbf{P}^{\uparrow}_{k+1}(x)+\alpha\mathbf{P}^{\downarrow}_{k-1}(x)] \tag{A8}$$

with the boundary conditions

$$\mathbf{P}^{\uparrow}_{a}(x)=1; \qquad \mathbf{P}^{\downarrow}_{0}=0. \tag{A9}$$

We may eliminate either $\mathbf{P}^{\uparrow}$ or $\mathbf{P}^{\downarrow}$ to give

$$\alpha x\mathbf{P}^{\uparrow}_{k+2}(x)-[1+(\alpha-\beta)x^2]\mathbf{P}^{\uparrow}_{k+1}(x)+\alpha x\mathbf{P}^{\uparrow}_{k}(x)=0 \tag{A10}$$

with an identical equation for $\mathbf{P}^{\downarrow}$. To solve this equation we look for a solution of the form $\mathbf{P}_k\sim w^k$, where w satisfies

$$\alpha x w^2(x)-[1+(\alpha-\beta)x^2]\,w(x)+\alpha x=0, \tag{A11}$$

which has two solutions

$$w_{\pm}(x)=\frac{[1+(\alpha-b)x^2]\pm\sqrt{[1+(\alpha-\beta)x^2]^2-4\alpha^2x^2}}{2\alpha x}. \tag{A12}$$

The general solution is

$$\mathbf{P}^{\uparrow}_{k}(x)=A'w^{k}_{+}(x)+B'w^{k}_{-}(x)$$
$$\mathbf{P}^{\downarrow}_{k}(x)=C'w^{k}_{+}(x)+D'w^{k}_{-}(x) \tag{A13}$$

The coefficients are determined by using the recurrence relations and the boundary conditions. Before doing this it is convenient to change variables from x to ϕ where

$$w_{\pm}=\mathrm{e}^{\pm i\phi}; \qquad \cos\phi\equiv\frac{[1+(\alpha-\beta)x^2]}{2\alpha x} \tag{A14}$$

so that

$$\mathbf{P}^{\uparrow}_{k}=A\sin k\phi+B\cos k\phi$$
$$\mathbf{P}^{\downarrow}_{k}=C\sin k\phi+D\cos k\phi. \tag{A15}$$

The boundary conditions lead to the two conditions

$$A\sin a\phi+B\cos a\phi=1; \qquad D=0. \tag{A16}$$

Taking the ratio between the two equations (A8) leads to a quadratic form in $\cos k\phi$ and $\sin k\phi$ which is zero. Therefore the coefficients are zero independent of k and this leads to the two conditions (where I have already put $D=0$)

$$A^2+B^2=C^2 \tag{A17}$$

$$\beta A \sin\phi+\beta B\cos\phi-\alpha C\sin\phi=0. \tag{A18}$$

From equation (A17) we may write $A=C\cos\delta$, $B=C\sin\delta$. Equation (A18) gives

$$\delta=\sin^{-1}\left(\frac{\alpha}{\beta}\sin\phi\right)-\phi \tag{A19}$$

and using the boundary condition gives the generating functions as

$$\mathbf{P}_k^{\uparrow}=\frac{\sin\{\sin^{-1}[(\alpha/\beta)\sin\phi]+(k-1)\phi\}}{\sin\{\sin^{-1}[(\alpha/\beta)\sin\phi]+(a-1)\phi\}} \tag{A20}$$

$$\mathbf{P}_k^{\downarrow}=\frac{1}{\sin\{\sin^{-1}[(\alpha/\beta)\sin\phi]+(a-1)\phi\}}. \tag{A21}$$

It is easily verified that this leads to the correct probabilities in the limit $\phi\to 0$ and $\alpha\to\beta$.

Appendix B: Calculation of temperature properties

To generate the temperature distributions shown in the figures, a sample of random half-walks was constructed. These start from halfway between the plates (in terms of steps, for a variable step size) and equal numbers of half-walks reach each plate. This allows a much larger sample of full walks to be constructed from pairs of half walks. The temperature at the join is given a value $T_{1/2}$, and the mean and standard deviation of this are easily calculated.

The temperature varies linearly (from either 0 to $T_{1/2}$ or from $T_{1/2}$ to 1) along the half-walks. The properties of the temperature distribution at a point are calculated from the distribution of temperatures on the random walks that pass through that point in terms of $T_{1/2}$. The distribution of temperatures $T_{1/2}$ is then included.

For Fig. 2 a spherical system is modelled by increasing the step length with radius, so that the step length at radius r is a certain fraction of r (0.2 in this case). Two- or three-dimensional models can be constructed by allowing the walk to be parallel to the plates. The effect of this was found to be negligible.