Hyperbolic Fixed Points, Topological Equivalence, and Structural Stability

If $\mathrm{Re}(\lambda) \neq 0$ for both eigenvalues, the fixed point is often called ***hyperbolic***. (This is an unfortunate name—it sounds like it should mean "saddle point"—but it has become standard.) Hyperbolic fixed points are sturdy; their stability type is unaffected by small nonlinear terms. Nonhyperbolic fixed points are the fragile ones.

We've already seen a simple instance of hyperbolicity in the context of vector fields on the line. In Section 2.4 we saw that the stability of a fixed point was accurately predicted by the linearization, *as long as* $f'(x^*) \neq 0$. This condition is the exact analog of $\mathrm{Re}(\lambda) \neq 0$.

These ideas also generalize neatly to higher-order systems. A fixed point of an nth-order system is *hyperbolic* if all the eigenvalues of the linearization lie off the imaginary axis, i.e., $\mathrm{Re}(\lambda_i) \neq 0$ for $i = 1, \ldots, n$. The important *Hartman–Grobman theorem* states that the local phase portrait near a hyperbolic fixed point is "topologically equivalent" to the phase portrait of the linearization; in particular, the stability type of the fixed point is faithfully captured by the linearization. Here ***topologically equivalent*** means that there is a ***homeomorphism*** (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time (the direction of the arrows) is preserved.

Intuitively, two phase portraits are topologically equivalent if one is a distorted version of the other. Bending and warping are allowed, but not ripping, so closed orbits must remain closed, trajectories connecting saddle points must not be broken, etc.

Hyperbolic fixed points also illustrate the important general notion of structural stability. A phase portrait is ***structurally stable*** if its topology cannot be changed by an arbitrarily small perturbation to the vector field. For instance, the phase portrait of a saddle point is structurally stable, but that of a center is not: an arbitrarily small amount of damping converts the center to a spiral.

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:

1. Each species would grow to its carrying capacity in the absence of the other. This can be modeled by assuming logistic growth for each species (recall Section 2.3). Rabbits have a legendary ability to reproduce, so perhaps we should assign them a higher intrinsic growth rate.

2. When rabbits and sheep encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually the sheep nudges the rabbit aside and starts nibbling (on the grass, that is). We'll assume that these conflicts occur at a rate proportional to the size of each population. (If there were twice as many sheep, the odds of a rabbit encountering a sheep would be twice as great.) Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

A specific model that incorporates these assumptions is

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

where

$x(t) =$ population of rabbits,
$y(t) =$ population of sheep

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

To find the fixed points for the system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. To classify them, we compute the Jacobian:

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Now consider the four fixed points in turn:

$$(0,0)\text{: Then } A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues are $\lambda = 3, 2$ so $(0,0)$ is an *unstable node.* Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $\mathbf{v} = (0,1)$, which spans the y-axis. (Recall the general rule: at a node, trajectories are tangential to the slow eigendirection, which is the eigendirection with the smallest $|\lambda|$.) Thus, the phase portrait near $(0,0)$ looks like Figure 6.4.1.

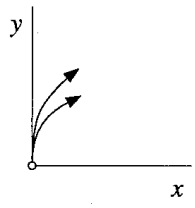

Figure 6.4.1

$$(0,2)\text{: Then } A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda = -1, -2$, as can be seen from inspection, since

the matrix is triangular. Hence the fixed point is a *stable node.* Trajectories approach along the eigendirection associated with $\lambda = -1$; you can check that this direction is spanned by $\mathbf{v} = (1, -2)$. Figure 6.4.2 shows the phase portrait near the fixed point $(0, 2)$.

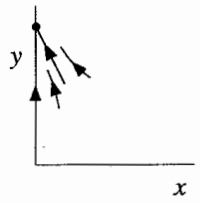

Figure 6.4.2

$(3, 0)$: Then $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ and $\lambda = -3, -1$.

This is also a *stable node.* The trajectories approach along the slow eigendirection spanned by $\mathbf{v} = (3, -1)$, as shown in Figure 6.4.3.

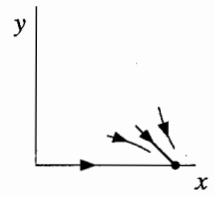

Figure 6.4.3

$(1, 1)$: Then $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, which has $\tau = -2$, $\Delta = -1$, and $\lambda = -1 \pm \sqrt{2}$. Hence this is a *saddle point.* As you can check, the phase portrait near $(1, 1)$ is as shown in Figure 6.4.4.

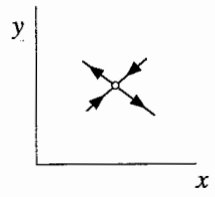

Figure 6.4.4

Combining Figures 6.4.1–6.4.4, we get Figure 6.4.5, which already conveys a good sense of the entire phase portrait. Furthermore, notice that the x and y axes contain straight-line trajectories, since $\dot{x} = 0$ when $x = 0$, and $\dot{y} = 0$ when $y = 0$.

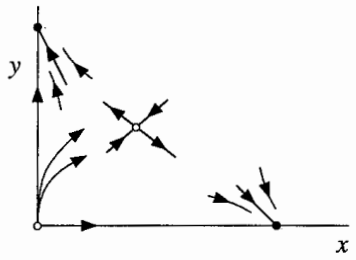

Figure 6.4.5

Now we use common sense to fill in the rest of the phase portrait (Figure 6.4.6). For example, some of the trajectories starting near the origin must go to the stable node on the x-axis, while others must go to the stable node on the y-axis. In between, there must be a special trajectory that can't decide which way to turn, and so it dives into the saddle point. This trajectory is part of the ***stable manifold*** of the saddle, drawn with a heavy line in Figure 6.4.6.

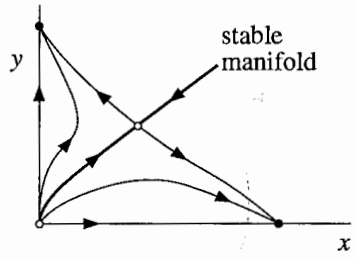

Figure 6.4.6

The other branch of the stable manifold consists of a trajectory coming in "from infinity." A computer-generated phase portrait (Figure 6.4.7) confirms our sketch.

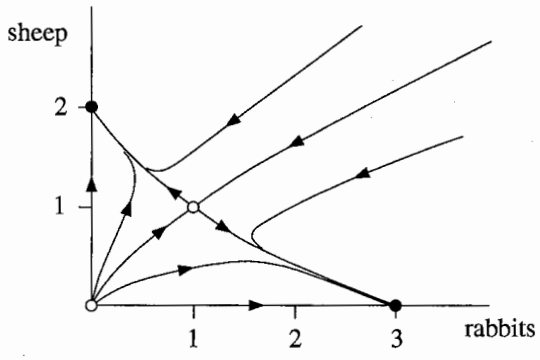

Figure 6.4.7

The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. This dichotomy occurs in other models of competition and has led biologists to formulate the *principle of competitive exclusion*, which states that two species competing for the same limited resource typically cannot coexist. See Pianka (1981) for a biological discussion, and

Pielou (1969), Edelstein–Keshet (1988), or Murray (1989) for additional references and analysis.

Our example also illustrates some general mathematical concepts. Given an attracting fixed point $\mathbf{x}^*$, we define its ***basin of attraction*** to be the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. For instance, the basin of attraction for the node at $(3,0)$ consists of all the points lying below the stable manifold of the saddle. This basin is shown as the shaded region in Figure 6.4.8.

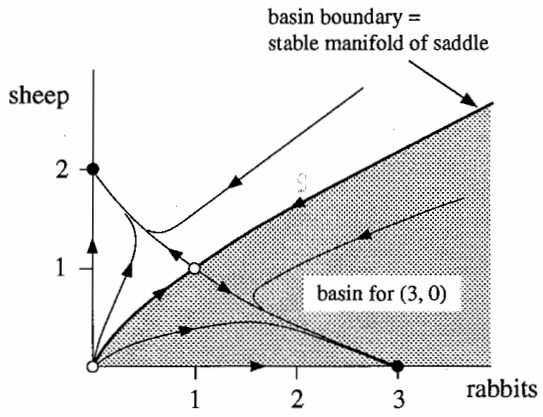

Figure 6.4.8

Because the stable manifold separates the basins for the two nodes, it is called the ***basin boundary***. For the same reason, the two trajectories that comprise the stable manifold are traditionally called ***separatrices***. Basins and their boundaries are important because they partition the phase space into regions of different long-term behavior.

6.5 Conservative Systems

Newton's law $F = ma$ is the source of many important second-order systems. For example, consider a particle of mass m moving along the x-axis, subject to a nonlinear force $F(x)$. Then the equation of motion is

$$m\ddot{x} = F(x).$$

Notice that we are assuming that F is independent of both $\dot{x}$ and t; hence there is no damping or friction of any kind, and there is no time-dependent driving force.

Under these assumptions, we can show that *energy is conserved*, as follows. Let $V(x)$ denote the ***potential energy***, defined by $F(x) = -dV/dx$. Then

$$m\ddot{x} + \frac{dV}{dx} = 0. \qquad (1)$$