# Computational Astrophysics 8

# The Equations of Magneto-Hydrodynamics

Romain Teyssier

Oscar Agertz

University of Zurich

# Outline

- Maxwell's equation
- Particle trajectories
- Kinetic theory
- Ideal MHD equations
- MHD waves
- Ambipolar diffusion

# Maxwell's equation for the electromagnetic field

Gauss's laws

$$\nabla \cdot \mathbf{E} = 4\pi q$$

$$\nabla \cdot \mathbf{B} = 0$$

No magnetic monopole

Ampère's law

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{j}$$

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

Classical theory for the electromagnetic field

# Maxwell's equation in integral form

Consider a volume V bounded by a surface S:

Gauss theorem
$$\int_S \mathbf{E} \cdot \mathbf{n} \mathrm{d}^2 S = 4\pi \int_V q \mathrm{d}^3 V$$

Flux conservation
$$\int_S \mathbf{B} \cdot \mathbf{n} \mathrm{d}^2 S = 0$$

Consider a surface S bounded by a contour L:

$$\int_L \mathbf{E} \cdot \mathbf{n} \mathrm{d} L = -\frac{1}{c} \partial_t \int_S \mathbf{B} \cdot \mathbf{n} \mathrm{d}^2 S$$

$$\int_L \mathbf{B} \cdot \mathbf{n} \mathrm{d} L = 4\pi \int_S \mathbf{j} \cdot \mathbf{n} \mathrm{d}^2 S + \frac{1}{c} \partial_t \int_S \mathbf{E} \cdot \mathbf{n} \mathrm{d}^2 S$$

# Related conservation laws

If one takes the divergence of Faraday's law (induction equation):

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\frac{1}{c}\partial_t \nabla \cdot \mathbf{B} = 0$$

If initially one has $\nabla \cdot \mathbf{B} = 0$ then this property is conserved for t>0.

If one takes the divergence of Ampère's law:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \frac{1}{c}\partial_t \nabla \cdot \mathbf{E} + 4\pi \nabla \cdot \mathbf{j} = 0$$

Using Gauss theorem, we get the charge conservation equation:

$$\frac{1}{c}\partial_t q + \nabla \cdot \mathbf{j} = 0$$

# A simple case: the Hydrogen plasma

$n_e$ electron density $e = 4.8032 \times 10^{-10}$ Coulomb

$n_n$ neutral density or neutral Hydrogen density

$n_i$ ion density or proton density or ionized Hydrogen density

Proton mass $m_p = 1.6726 \times 10^{-24}$ g

Electron mass $m_e = 9.1094 \times 10^{-28}$ g

Total mass density

$$\rho = n_e m_e + n_i m_p + n_n m_H \simeq (n_i + n_n) m_p$$

Total charge density $q = e(n_i - n_e)$

Total current density $\mathbf{j} = \frac{e}{c}(n_i \mathbf{v}_i - n_e \mathbf{v}_e)$

# Particle motions in a constant electromagnetic field

Particle acceleration due to the Lorentz force:

$$m\frac{d\mathbf{v}}{dt} = \pm e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$$

We are looking for solutions of the form: $\mathbf{v}(t) = \mathbf{v}_d + \mathbf{v}_g(t)$

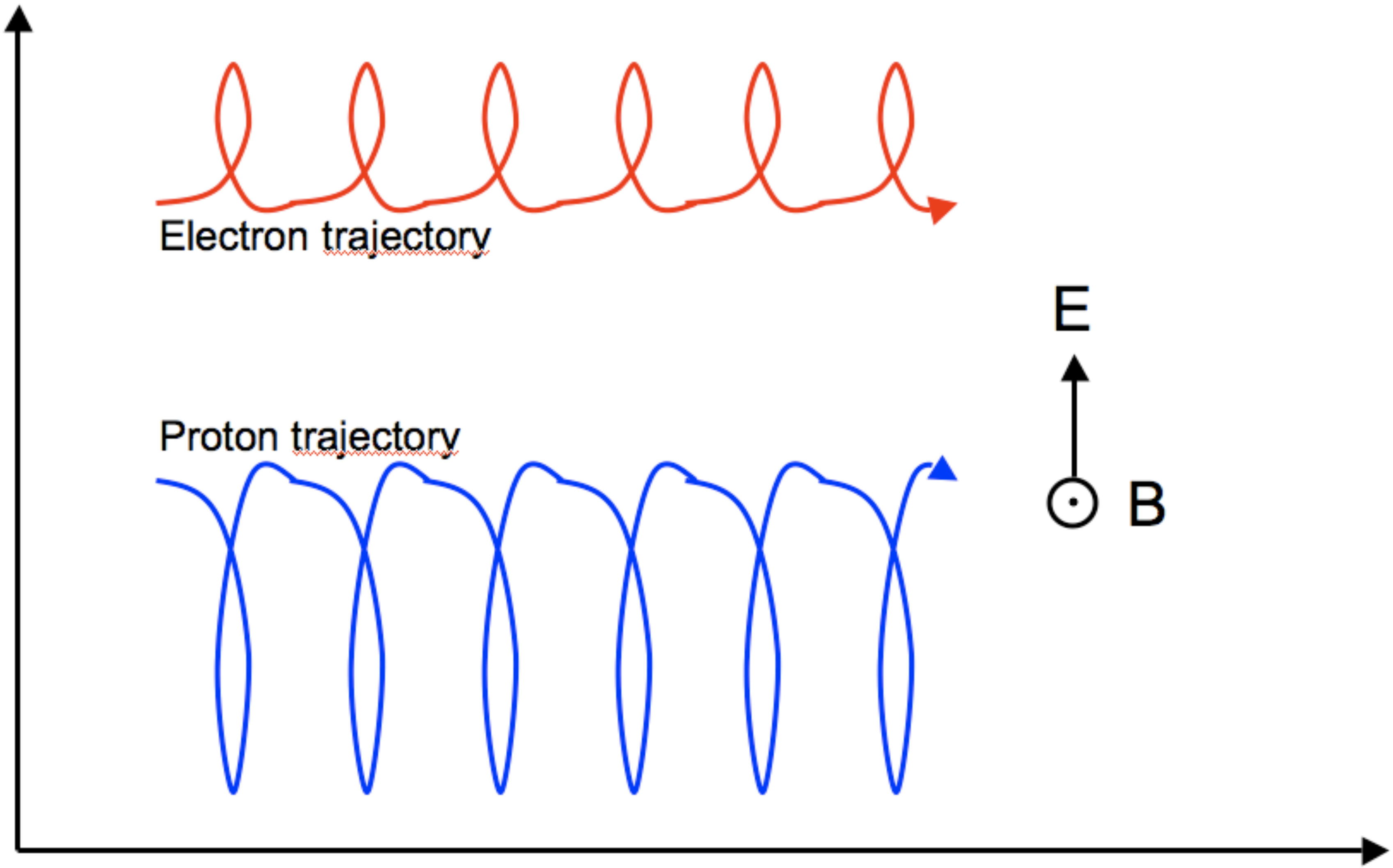
Drift motion with constant velocity $\mathbf{v}_d = c\frac{\mathbf{E}\times\mathbf{B}}{B^2}$ $\quad E_{//} = 0$

Gyro motion with time-varying velocity $m\frac{d\mathbf{v}_g}{dt} = \frac{\pm e}{c}\mathbf{v_g}\times\mathbf{B}$

Equilibrium between centrifugal and Lorentz forces gives

$m\frac{{v_g}^2}{r} = \frac{e}{c}v_g B$ Electron and proton gyroradius $r_g = \frac{m v_g c}{eB}$

# Particle motions in a constant electromagnetic field



# Kinetic theory for a magnetized plasma

Electron, proton and neutral distribution fonctions

$$n_e = \int f_e(\mathbf{x}, \mathbf{v}, t) d^3v \quad n_i = \int f_i(\mathbf{x}, \mathbf{v}, t) d^3v \quad n_n = \int f_n(\mathbf{x}, \mathbf{v}, t) d^3v$$

Boltzmann equation with Lorentz force

$$\partial_t f + \dot{\mathbf{x}}\partial_{\mathbf{x}} f + \dot{\mathbf{v}}\partial_{\mathbf{v}} f = (\partial_t f)_{coll}$$

$$\dot{\mathbf{v}} = \pm \frac{e}{m}(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$$

$$\dot{\mathbf{x}} = \mathbf{v}$$

Two limiting cases:

1- Fully ionized Hydrogen plasma

2- Quasi-neutral fluid

# Fully ionized Hydrogen plasma

Electron momentum conservation equation, using the Relaxation Time Approximation for the collision term:

$$\partial_t(n_e m_e \mathbf{u}_e) + \partial_x(n_e m_e \mathbf{u}_e \mathbf{u}_e) + \partial_x \mathbf{P}_e + n_e e(\mathbf{E} + \frac{1}{c}\mathbf{u}_e \times \mathbf{B}) = -n_e m_e \frac{\mathbf{u}_e - \mathbf{u}_i}{\tau_{ei}}$$

Using the current density, we can write the collision term as: $\frac{m_e c}{n_e \tau e}\mathbf{j}$

Because of the small electron mass, we neglect electron momentum and therefore we get the following equilibrium current:

$$\eta \mathbf{j} = \frac{1}{n_e e}\partial_x \mathbf{P}_e + (\mathbf{E} + \frac{\mathbf{u}_e}{c} \times \mathbf{B}) \qquad \eta = \frac{m_e c}{n_e \tau e^2}$$

where we introduce the plasma resistivity.

For a fully ionized Hydrogen plasma, $\tau \propto \frac{T_e^{3/2}}{n_e}$ so that $\eta \propto T_e^{-3/2}$

Resistivity does not depend on density and is larger for a cold plasma.

# Fully ionized Hydrogen plasma

Ion momentum conservation equation with the opposite collision term (ion-electron collisions conserve total momentum):

$$\partial_t(n_i m_i \mathbf{u}_i) + \partial_x(n_i m_i \mathbf{u}_i \mathbf{u}_i) + \partial_x \mathbf{P}_i - n_i e(\mathbf{E} + \frac{1}{c}\mathbf{u}_i \times \mathbf{B}) = +n_e m_e \frac{\mathbf{u}_e - \mathbf{u}_i}{\tau_{ei}}$$

Using the equilibrium current in the collision term, we get the momentum equation:

$$\partial_t(\rho \mathbf{u}) + \partial_x(\rho \mathbf{u}\mathbf{u}) + \partial_x(\mathbf{P}_e + \mathbf{P}_i) - \mathbf{j} \times \mathbf{B} = 0$$

where mass is carried mainly by ions so that $\rho \simeq n_i m_i \quad \mathbf{u} \simeq \mathbf{u}_i$

The electric field writes: $\mathbf{E} \simeq -\frac{\partial_x P_e}{n_e e} - \frac{\mathbf{u}}{c} \times \mathbf{B} + \eta \mathbf{j}$

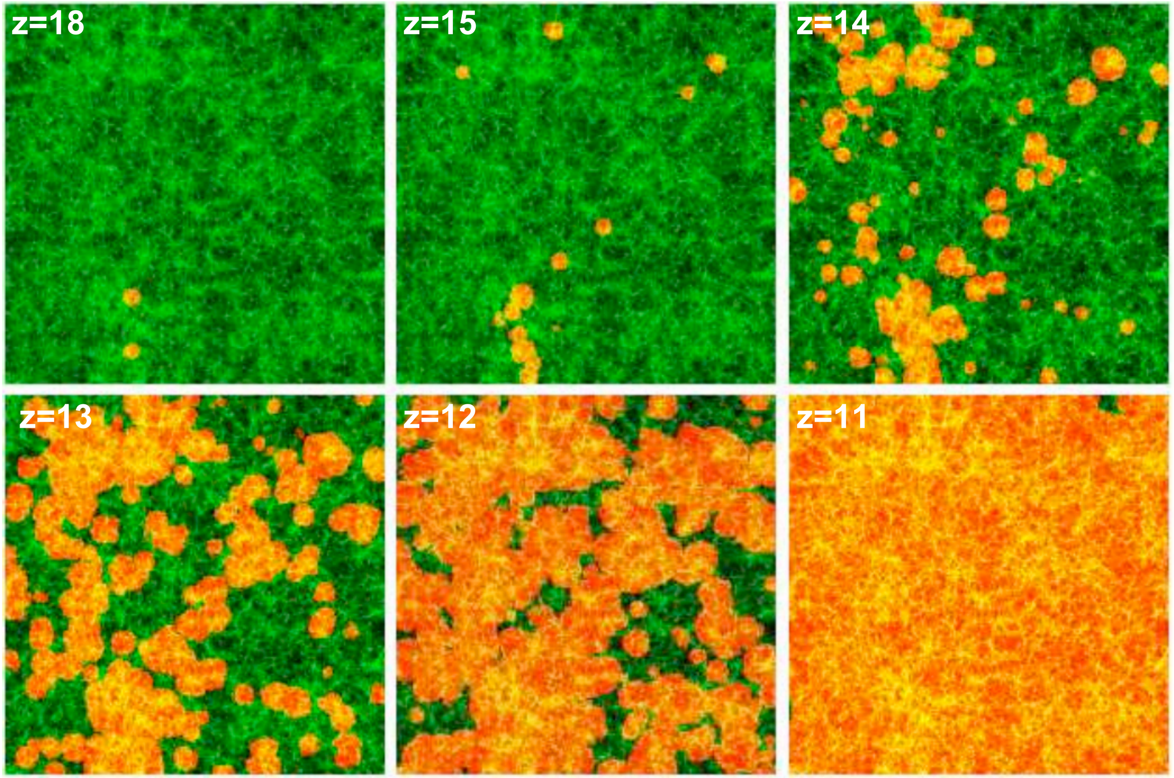
so that Faraday's law becomes:

$$\partial_t \mathbf{B} = \frac{c}{e n_e^2}(\partial_x P_e) \times (\partial_x n_e) + \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (c \eta \mathbf{j})$$

Biermann battery | induction | resistivity

# Weak magnetic field limit

During the reionization epoch, young stars photo-ionize their surroundings. Green is neutral hydrogen; orange is ionized hydrogen and electrons.




The Biermann battery proceed because electrons pressure gradient induce microscopic electric fields and currents. Magnetic fields of $10^{-20}$-$10^{-18}$ Gauss are generated during the 100 Myrs of reionization.

# Ideal MHD limit

The magnetic field is strong enough so we can neglect the Biermann battery term, if the electron gyroradius is much smaller than the system size.

Exercise: prove this assuming that the flow velocity is sonic.

The magnetic resistivity is small enough so we can neglect the current term.

We get for the electric field $\mathbf{E} \simeq -\frac{\mathbf{u}}{c} \times B$

so that Faraday's law becomes the induction equation $\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B})$

Ampère's law writes $\nabla \times \mathbf{B} = 4\pi(\mathbf{j} + \mathbf{j}_d)$

where we neglect the displacement current $4\pi \mathbf{j}_d = \frac{1}{c}\partial_t \mathbf{E}$

Exercise: prove that the displacement current is negligible for a non relativistic flow.

# The ideal MHD equations

Mass conservation $$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation $$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + P) - \mathbf{j} \times \mathbf{B} = 0$$

Energy equation $$\partial_t (\rho \epsilon) + \nabla \cdot (\rho \epsilon \mathbf{u}) + P \nabla \cdot \mathbf{u} = 0$$

Induction equation $$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

Ampère’s law $$\nabla \times \mathbf{B} = 4\pi \mathbf{j}$$

No magnetic monopoles $$\nabla \cdot \mathbf{B} = 0$$

# The ideal MHD equations in conservative forms

Mass conservation
$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation
$$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} - \frac{1}{4\pi}\mathbf{B}\mathbf{B}) + \nabla P_{tot} = 0$$

Total energy conservation
$$\partial_t E + \nabla \cdot \left[ (E + P_{tot})\mathbf{u} - \frac{1}{4\pi}\mathbf{B}(\mathbf{B} \cdot \mathbf{u}) \right] = 0$$

Magnetic flux conservation
$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0$$

Total energy
$$E = \rho \epsilon + \frac{1}{2}\rho \mathbf{u}^2 + \frac{1}{8\pi}\mathbf{B}^2$$

Total pressure
$$P_{tot} = P + \frac{1}{8\pi}\mathbf{B}^2$$

No magnetic monopoles
$$\nabla \cdot \mathbf{B} = 0$$

# The ideal MHD equations in 1D

For 1D (plane symetric flow), one has $\nabla \cdot \mathbf{B} = 0 \rightarrow B_x = \text{constant}$

The vector of conservative variables writes $\mathbf{U} = (\rho, \rho u, \rho v, \rho w, B_y, B_z, E)$

Ideal MHD in conservative form: $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0$

Flux function

$$\mathbf{F} = \left| \begin{array}{l} \rho u \\ \rho u^2 + P_{tot} - B_x^2 \\ \rho u v - B_x B_y \\ \rho u w - B_x B_z \\ B_y u - B_x v \\ B_z u - B_x w \\ (E + P_{tot}) u - B_x (B_x u + B_y v + B_z w) \end{array} \right.$$

# MHD waves

Compute the Jacobian matrix $\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$

It has 7 real eigenvalues (ideal MHD equations are hyperbolic), one for each wave:

2 fast magnetosonic waves: $\lambda_1 = u - c_f$ $\lambda_7 = u + c_f$

2 Alfven waves: $\lambda_2 = u - c_a$ $\lambda_6 = u + c_a$

2 slow magnetosonic waves: $\lambda_3 = u - c_s$ $\lambda_5 = u + c_s$

1 entropy waves: $\lambda_4 = u$

Fast magnetosonic waves are longitudinal waves with variations in pressure and density (correlated with magnetic field)

Slow magnetosonic waves are longitudinal waves with variations in pressure and density (anti-correlated with magnetic field)

Alfven waves are transverse waves with no variation in pressure and density.
Entropy wave is a contact discontinuity with no variation in pressure and velocity.

# MHD wave speed

Sound speed: $c_0^2 = \frac{\gamma P}{\rho}$ Alfven speed: $c_{a,x}^2 = \frac{B_x^2}{4\pi\rho}$ $c_a^2 = \frac{B^2}{4\pi\rho}$

Fast magnetosonic speed: $c_f^2 = \frac{1}{2}(c_0^2 + c_a^2) + \frac{1}{2}\sqrt{(c_0^2 + c_a^2)^2 - 4c_0^2c_{a,x}^2}$

Slow magnetosonic speed: $c_s^2 = \frac{1}{2}(c_0^2 + c_a^2) - \frac{1}{2}\sqrt{(c_0^2 + c_a^2)^2 - 4c_0^2c_{a,x}^2}$

$$u - c_f < u - c_a < u - c_s < u < u + c_f < u + c_a < u + c_s$$

In some special cases, some wave speeds can be equal:

**The ideal MHD system is not strictly hyperbolic.**

This can lead to exotic features such as *compound waves*

(for example a mixture of Alfven wave and shock)

# Ideal MHD as a frozen-field limit

Integral form of the induction equation: $\partial_t \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{s} = \int_L (\mathbf{u} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l}$

For any surface bounded by a closed contour, the magnetic flux variation is given by the circulation of the electric field along the closed contour.

For any volume bounded by a closed surface, the total magnetic flux is zero:

$$\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{s} = \int_V (\nabla \cdot \mathbf{B}) \mathrm{d}v = 0$$

The induction equation writes $\partial_t \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B} - (\nabla \cdot \mathbf{u})\mathbf{B}$

Using the continuity equation, we get finally: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) = \left[\left(\frac{\mathbf{B}}{\rho}\right) \cdot \nabla\right]\mathbf{u}$

Defining a field line as $\delta\mathbf{x} \propto \mathbf{B}/\rho$ we get $\frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathbf{x}) = [(\delta\mathbf{x}) \cdot \nabla]\,\mathbf{u}$

The magnetic field lines are moving with the plasma velocity.

# Beyond ideal MHD : magnetic diffusion

If we don't neglect magnetic resistivity, we have $\mathbf{E} = \eta \mathbf{j} - \frac{\mathbf{u}}{c} \times B$

The induction equation now becomes

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{c}{4\pi} \nabla \times (\eta \nabla \times \mathbf{B})$$

Assuming a constant coefficient, we can write the resistive term as a diffusion process:

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{c\eta}{4\pi} \Delta \mathbf{B}$$

*D* is the magnetic diffusivity coefficient and $R_M$ the magnetic Reynolds number

$$D = \frac{c\eta}{4\pi} \qquad R_M = \frac{L\mathrm{u}}{D}$$

For a fully ionized Hydrogen plasma, one has $D=10^{13.1}\ T^{-3/2}$ in cgs units.

Microscopic magnetic diffusion is relevant for molecular core and planet formation.

# The diffusive MHD equations

Mass conservation $$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation $$\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + P) - \mathbf{j} \times \mathbf{B} = 0$$

Energy equation $$\partial_t(\rho \epsilon) + \nabla \cdot (\rho \epsilon \mathbf{u}) + P \nabla \cdot \mathbf{u} = c\eta \mathbf{j}^2$$

Induction equation $$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) - c\nabla \times (\eta \mathbf{j})$$

Ampère's law $$\nabla \times \mathbf{B} = 4\pi \mathbf{j}$$

No magnetic monopoles $$\nabla \cdot \mathbf{B} = 0$$

# Quasi-neutral fluid: ambipolar diffusion

We write the 3 momentum conservation laws with the main collision terms:

$$\partial_t(n_e m_e \mathbf{u}_e) + \partial_x(n_e m_e \mathbf{u}_e \mathbf{u}_e) + \partial_x \mathbf{P}_e + n_e e(\mathbf{E} + \frac{1}{c}\mathbf{u}_e \times \mathbf{B}) = -n_e m_e \frac{\mathbf{u}_e - \mathbf{u}_i}{\tau_{ei}}$$

$$\partial_t(n_i m_i \mathbf{u}_i) + \partial_x(n_i m_i \mathbf{u}_i \mathbf{u}_i) + \partial_x \mathbf{P}_i - n_i e(\mathbf{E} + \frac{1}{c}\mathbf{u}_i \times \mathbf{B}) = +n_e m_e \frac{\mathbf{u}_e - \mathbf{u}_i}{\tau_{ei}} - n_i m_i \frac{\mathbf{u}_i - \mathbf{u}_n}{\tau_{in}}$$

$$\partial_t(n_n m_n \mathbf{u}_n) + \partial_x(n_n m_n \mathbf{u}_n \mathbf{u}_n) + \partial_x \mathbf{P}_n = +n_i m_i \frac{\mathbf{u}_i - \mathbf{u}_n}{\tau_{in}}$$

We neglected electron (versus ion) momentum because $m_e << m_i$.

Now, we also neglect ion (versus neutral) momentum because $n_i << n_n$.

Neglecting ion and electron pressures, we get $\mathbf{j} \times \mathbf{B} = n_i m_i \frac{\mathbf{u}_i - \mathbf{u}_n}{\tau_{in}}$

We finally get for the neutral fluid the ideal MHD equations

$$\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + P) - \mathbf{j} \times \mathbf{B} = 0$$

$$\mathbf{u} \simeq \mathbf{u}_n$$

$$\rho \simeq n_n m_n$$

# Quasi-neutral fluid: ambipolar diffusion

The induction equation still writes: $$\partial_t \mathbf{B} = \nabla \times (\mathbf{u}_i \times \mathbf{B})$$

Using the equilibrium (force balance) limit, $$\mathbf{u}_i = \mathbf{u}_n + \frac{\tau_{in}}{\rho_i}\mathbf{j} \times \mathbf{B}$$

We get the induction equation in the ambipolar limit

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla \times (\mathbf{v}_d \times \mathbf{B})$$

$\mathbf{v}_d$ drift velocity

Using Ampère's law, $\nabla \times \mathbf{B} = 4\pi \mathbf{j}$ we get a very complex, non-linear diffusion process for the field lines with respect to the flow (ion) velocity that writes

$$\nabla \times \left[\left(\frac{\tau_{in}}{4\pi\rho_i}[\nabla \times \mathbf{B}] \times \mathbf{B}\right) \times \mathbf{B}\right] \qquad \tau_{in} = \frac{10^{-13}}{\rho_n} \text{ sec}$$

By analogy to magnetic diffusion, one can define the diffusion coefficient $D$

$$D \simeq 10^{-13}\frac{B^2}{4\pi\rho_i\rho_n}$$

and the ambipolar diffusion time scale $t_{AD} = L^2/D$

# Conclusion

- Kinetic theory as the origin of the ideal MHD equations

  1- we neglected electrons inertia,

  2- we neglected displacement current,

  3- we assume vanishing resistivity (small collision time)

- Non-ideal effects that can be taken into account easily:

  1- by adding resistivity

  2- by considering ambipolar diffusion (quasi-neutral fluid)

  3- by adding viscosity and thermal conduction

- Transport coefficients (viscosity and thermal conduction) now depend on the direction and on the magnitude of the magnetic field (see Braginskii, S.I., 1965, Rev. Plasma Phys.)

**Next lecture: computational MHD**