

### 3. The Electric Flux

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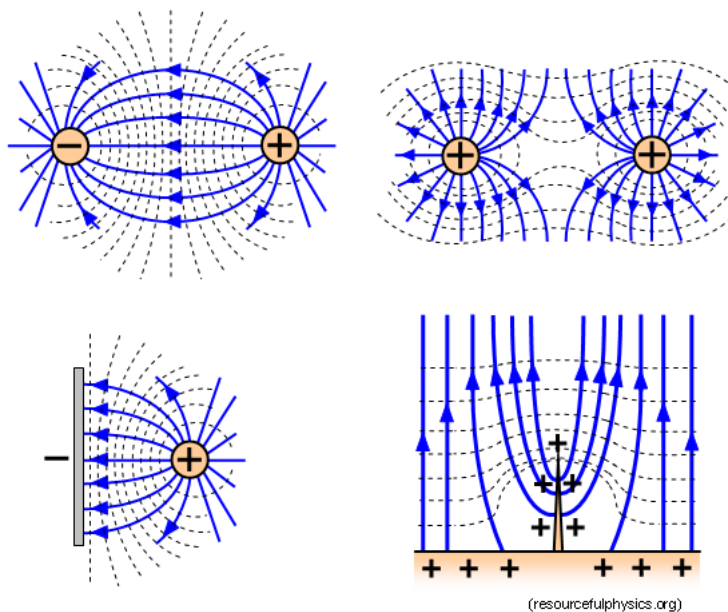
#### 1 Electric Field Lines

The most important idea so far is the *electric field*. If you place a charge at any point, the electric force it feels is proportional to its charge. Thus the force divided by the charge is the same for all particles at that point. This quantity is the electric field at that point. It is a vector  $\mathbf{E}$  : the direction of the electric field is the same as the direction of the force felt by a *positive* charge placed at that point. A negative charge would be pushed in the direction opposite to the electric field.

$$\mathbf{F} = q\mathbf{E}.$$

Thus the electric field created by a positive charge points outward along the radial lines from its position; for a negative charge, the field points towards its position.

It is useful to think of an electric field at each point as a little arrow at that point, giving its direction. Imagine moving along a curve that at each point points in the direction of the electric field. This is called a *field line*. These line start at a positive charge and end at a negative charges. They cross only at points where you don't know the direction of the electric field: where the electric field vanishes. Or at the location of point charges where the electric field is infinite.



## 2 The Path of a Charged Particle

The field lines are *not* the path of a charged particle in the electric field. It gives the direction of the force- and hence the acceleration- at each point: the direction in which a particle moves is given by the velocity. Velocity and acceleration don't have to point in the same direction.

For example, a negative charge can move in a circle around a positive charge which is at rest. Its acceleration points to the positive charge, although its velocity points tangential to the circle.

### 2.1 A Charged Particle in a Constant Electric Field

The path of a charged particle in a constant electric field is a parabola. This is very much like the path of a ball you throw into the air: the gravitational force is constant, so the acceleration is constant.

If the electric field  $\mathbf{E}$  is constant, the acceleration is constant as well:

$$\mathbf{a} = \frac{q}{m}\mathbf{E}.$$

The velocity is then a linear function of time:

$$\mathbf{v} = \mathbf{a}t + \mathbf{v}_0$$

where  $\mathbf{v}_0$  is the velocity at time  $t = 0$ . The position at any time  $t$  must be a function whose derivative is equal to velocity

$$\mathbf{r}(t) = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0t + \mathbf{r}_0$$

where  $\mathbf{r}_0$  is the position at time  $t = 0$ . This is the equation of a parabola. Every shape (curve) can be described by a picture or an equation. The equation is by far the more powerful way of thinking of about it: it tells you much more about the curve than a mere picture.

*They say a picture is worth a thousand words.*

*I say an equation is worth a thousand pictures.*

I can differentiate and integrate an equation. Try doing that with a picture.

### 3 The Electric Flux

Karl Friedrich Gauss is one of the greatest mathematicians of all time. Some would say he is the greatest. (Not me. That would be Isaac Newton.) Gauss found a way of thinking about Coulomb's law that allows us to calculate the electric field of charge distributions in a simple way. Although Gauss did not discover a new law of physics, his restatement of Coulomb's law was just as important a contribution.

To understand Gauss's law we need to master the idea of the Electric Flux across a surface. It is a way of measuring the total electric field that crosses a surface.

#### 3.1 Constant Field and Planar Surface

Again we start with the simplest situation: the field is a constant and the surface is part of a plane perpendicular to it. The electric flux across the surface is defined to be the area of the surface  $A$  times the electric field.

$$\Phi = EA$$

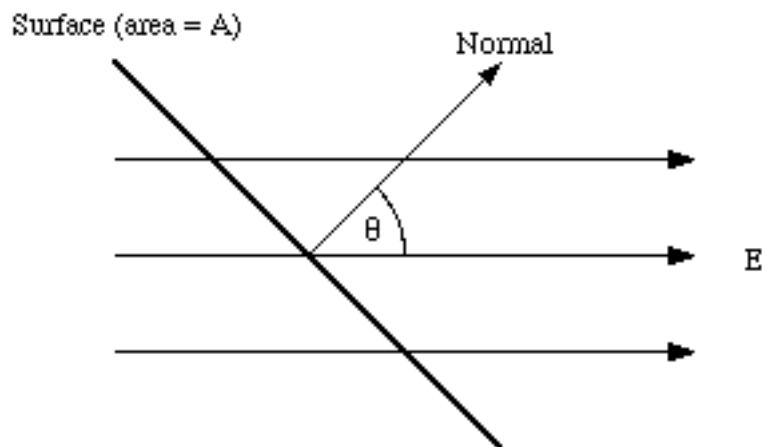
In this situation the electric field lines cross the surface: the flux is a way of counting how many field lines cross the surface. If we tilt the surface so that the lines cross at an angle, we can imagine fewer of them cross it. If the surface is lying parallel to the lines, nothing crosses it.

The flux crossing a surface whose normal is at an angle  $\theta$  to the field lines is defined to be

$$\Phi = EA \cos \theta$$

Every surface has a direction associated to it, which is its normal. The area of a surface can be thought of as a vector, whose direction is along the normal. The flux is the dot product of the electric field and this area vector.

$$\Phi = \mathbf{E} \cdot \mathbf{A}.$$



### 3.2 Curved Surfaces

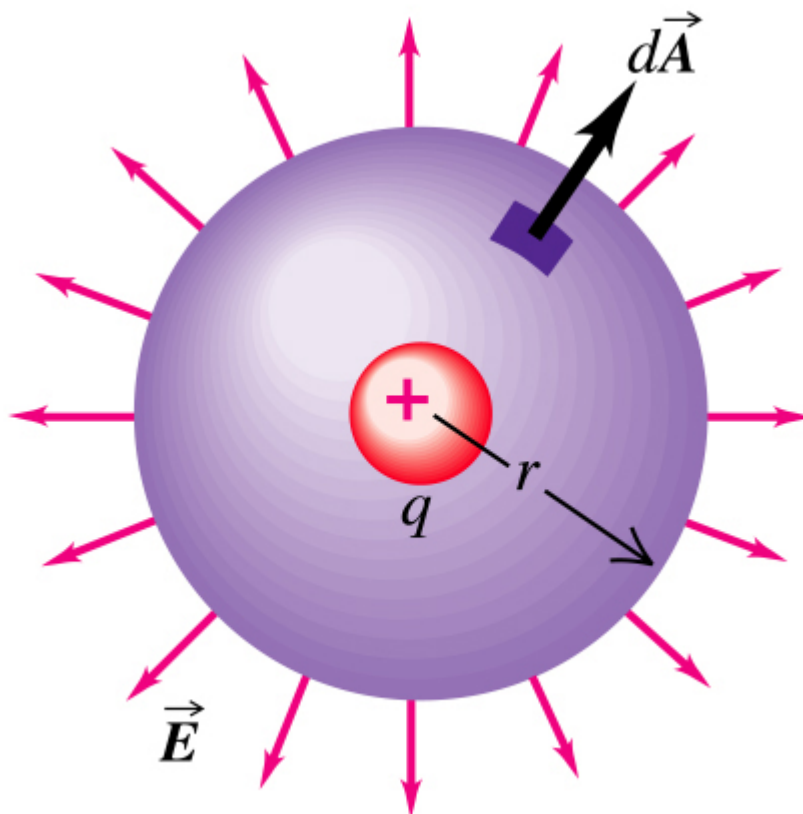
Alas, most surfaces we encounter are not planes. Here is where the ideas of calculus come in handy. We can think of any smooth surface as tiled by small flat pieces: like the roof a house. If the tiles are small enough, this becomes a better and better way to think of the surface. Each tile has a small area and its normal points in a different direction. We can calculate the flux through each tile and then add them all up to get the total flux through the surface. This is called a *surface integral*. It is denoted by

$$\Phi = \int \mathbf{E} \cdot d\mathbf{A}$$

#### 3.2.1 Flux of a Point Charge

As an example, let us find the flux through a sphere surrounding a point charge. The electric field is

$$\mathbf{E} = k \frac{Q}{r^2} \hat{\mathbf{r}}.$$



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A little piece of the sphere will always have a normal that points radially outward: that is a special property of the sphere. So we are in luck: although the normal to the sphere and the electric have directions that change with position, they are always parallel in this case. Moreover, the magnitude of the electric field is a constant on the sphere. So the total flux in this case is just the magnitude of the electric field at a sphere of radius  $r$  times the area of the sphere of radius  $r$ .

$$\Phi = k \frac{Q}{r^2} 4\pi r^2$$

This is the same for all spheres centered at the charge, the radius actually cancels out:

$$\Phi = 4\pi kQ.$$

In other words

$$Q = \frac{\Phi}{4\pi k}$$

### 3.3 An Annoying Constant

The constant  $\frac{1}{4\pi k}$  appears very often in this subject, so it has a standard name. It is called the *permittivity of free space* (don't ask) and is denoted by the symbol

$$\epsilon_0 = \frac{1}{4\pi k}.$$

Its value is  $8.9 \times 10^{-12} C^2 N^{-1} m^{-2}$ .

The total flux out of a point charge  $Q$  is

$$\Phi = \frac{Q}{\epsilon_0}.$$

### 3.4 The Direction of the Normal

The area of a surface points along its normal. If a surface is closed (is the boundary of a volume) we always choose this direction to be the *outward normal*. The total flux out of a closed surface is denoted by  $\oint$ : the little loop is to remind us that we are integrating over a closed surface.

## 4 Gauss's Law

Gauss realized that Coulomb's law can be stated in another way:

The electric flux through any closed surface is proportional to the total charge contained inside it.

In other words

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0}.$$

We saw this for the simplest case of an electric charge and a sphere. It was Gauss's insight that this is also true of all surfaces and any electric field! We will find it enormously useful in calculating the electric field of many shapes of charges.

## References

- [1] We used diagrams from notes online by Profs. Frank Wolfs (UR), J. F. Becker (SJSU) and by resourcefulphysics.org