9. RC Circuits

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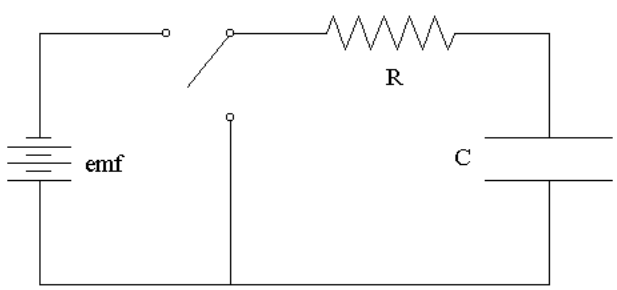
Most electrical devices have a source of energy (battery or power supply connected to an eletrical grid) that drives an electrical current through a complete loop or circuit. The simplest such device is a circuit consisting of a capacitor and a resistor . We can also add a switch that connects the device to a battery so that the capacitor can be charged; or allow it the capacitor to discharge bypassing the battery. If there were no resistance, the capacitor would discharge instantaneously. But in reality, it takes a finite time that is proportional to the proportional to the product of resistance and capacitance.



1 Discharging the Capacitor

Suppose that the capacitor has been charged so that each side carries an electrical charge Q_0. Remember that the total charge on a capacitor is zero; Q is the amount of charge that has been moved from one plate to the other. If we now connect the positive side of the capacitor to the negative side through a resistance R, a current will flow until there is no more charge left. We want to determine the charge $Q(t)$ on the capacitor at any instant t.

The potential difference between the two plates of a capacitor is

$$V(t) = \frac{Q(t)}{C}.$$

This will drive a current

$$I(t) = \frac{V(t)}{R} = \frac{Q(t)}{RC}.$$

But the current is simply the rate of transfer of the charge.

$$I(t) = -\frac{dQ}{dt}.$$

As the capacitor discharges, the current is in the direction that will decrease $Q(t)$.

Thus we get

$$\frac{dQ}{dt} = -\frac{Q}{RC}.$$

This is an example of a differential equation: the derivative of a quantity is related to itself.

The function whose derivative is equal to itself is the exponential.

$$\frac{d\ e^x}{dx} = e^x.$$

Notice also that

$$\frac{d\ e^{ax}}{dx} = ae^{ax}$$

for any constant a. Thus a solution to our equation is

$$Q(t) = e^{-\frac{t}{RC}}.$$

But it is not the only solution. If you multiply it by any constant you will get another solution. This is true of differential equations usually that they have many solutions and to pick the right one we need additional information. In our case, this is the knowledge of the charge at $t = 0$:

$$Q(0) = Q_0.$$

Thus

$$Q(t) = Q_0 e^{-\frac{t}{RC}}.$$

This tells you how fast the charge is discharged.

1.1 The Time Constant

The quantity

$$\tau = RC$$

that appears above has dimensions of time.

For a resistance of $R = 20k\Omega$, and a capacitance of $0.3\mu F$ (remember that a Farad is a very large unit of capacitance, so a micro-Farad is more the sort of capacitance you will encounter) the time constant is

$$\tau = 6ms$$

a few a milliseconds. This is about typical.

The smaller this time constant of the circuit the faster it will discharge. For example, the time time it takes to lose half the charge is given by

$$e^{-\frac{T}{\tau}} = \frac{1}{2}$$

or

$$T = \tau \log 2 \approx 0.69\tau.$$

This is called the half-life of the circuit. for the above example this is about $4ms$. In another $4ms$ the charge is reduced by half again; that is to a quarter of its original value. In $12ms$ it is reduced by another factor of two so that the charge an eighth of the original and so on. In principle, the charge never quite becomes zero: but in practice after a few half-lifes there is almost nothing left.

Such exponential decays are quite common in physics. If you have a radioactive material, the number of atoms that remain after time $t$is

$$N(t) = N_0 e^{-\frac{t}{\tau}}.$$

The time constant of the material again gives the half-life of the material as above.

2 Charging the Capacitor.

Suppose a capacitor has initial charge zero. If it is connected to a battery that supplies a constant potential difference $V_{0.}$,a charge will build up on the capacitor. The potential difference across the battery is equal in magnitude to the sum of the potential drop across the resistor and the capacitor

.

$$V_0 = RI(t) + \frac{Q(t)}{C}.$$

The current is in the direction that increases the charge:

$$I(t) = \frac{dQ}{dt}.$$

Thus we get another differential equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V_0.$$

This one is a bit harder because the r.h.s. is not zero. We already know the solution when $V_0 = 0$: it is proportional to $e^{-\frac{t}{RC}}$.

So let us turn the equation into what we know by rewriting it as

$$R\frac{dQ}{dt} + \frac{Q - CV_0}{C} = 0$$

If we define $Q_1(t) = Q(t) - CV_0$we get a familiar equation

$$R\frac{dQ_1}{dt} + \frac{Q_1}{C} = 0.$$

Thus

$$Q_1(t) = Ae^{-\frac{t}{\tau}}$$

$$Q(t) = CV_0 + Q_1(t) = CV_0 + Ae^{-\frac{t}{\tau}}.$$

We still have to determine A. In this case the initial charge is zero, $Q(t) = 0$.

$$CV_0 + A = 0$$

so that

$$Q(t) = CV_0 \left[1 - e^{-\frac{t}{\tau}}\right].$$

This shows a charge that starts zero and grows to a maximum value of CV_0.

3 Simplifying a Circuit

As long as we only have capacitors and resistors, any circuit can be reduced to the above simple one by repeated use of the rules for combining them in parallel and series. Or we can use Kirchhoff's rules. There are many examples in the end of the textbook.

By adding a switch that turns on when the voltage exceeds a critical value, we can produce a sawtooth waveform: the current grows to a maximum, then drops and starts to grow again. Most useful devices contain additional components such as inductors. We will study them later.

References

[1] The diagram is from http://biology.unm.edu/toolson