PHY 142 FINAL EXAM Saturday, December 19, 2009

Please put a box around around your final answer. Cross off any work you do not want the grader to consider.

Problem 1 [15 points]

Consider a sphere of radius R that contains a charge density $\rho(\mathbf{r}) = C\sqrt{r}$, where r is the radial distance from the center of the sphere and C is a positive constant.

a) [4 pts] What is the total charge contained in the sphere?

b) [4 pts] What is the electric field $\mathbf{E}(\mathbf{r})$ for $r < R$?

c) [4 pts] What is the electrostatic potential $\phi(\mathbf{r})$ for $r < R$? Choose your reference point such that $\phi(\mathbf{r}) \to 0$ as $r \to \infty$.

d) [3 pts] A positive point charge q is brought from infinitely far away and positioned at a distance $2R$ from the center of the sphere. How much work did it take to do this?

Problem 2 [15 points]

Consider two infinite flat planes that are parallel to each other and separated by a distance d. The normal vector to the planes points in the $\hat{z}$ direction. The top plane has a surface charge density $+\sigma$ while the bottom plane has a surface charge density of $-\sigma$.

a) [3 pts] What is the electric field between the planes?

b) [3 pts] What is the electric force per unit area between the two planes? Is it attractive or repulsive?

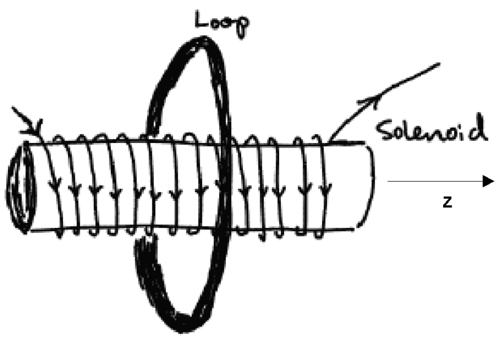
c) [3 pts] Now suppose the same charged planes are moving in the $\hat{x}$-direction with a constant velocity $\mathbf{v}$. Assume $v \ll c$ so relativistic effects need not be considered. What are the surface currents on the planes?

d) [3 pts] Find the magnetic field everywhere in space.

e) [3 pts] What is the magnetic force per unit area between the plates? Is it attractive or repulsive? How does it compare to the electric force?

Problem 3 [25 points total]

a) A circular wire loop of radius a is placed outside of and concentric with a cylindrical solenoid of radius b. The axis of the solenoid is in the $\hat{z}$ direction. The solenoid can be considered *long* (i.e. its length $L \gg b$). It has N turns per unit length and carries a steady current I.



i) [3 pts] What is the magnetic flux through the loop in the z direction?

ii) [2 pts] At time $t = 0$, the current I in the solenoid decreases linearly to half its initial value over a time interval Δt. What is the magnitude of the *emf* $\mathcal{E}$ induced around the loop?

iii) [3 pts] If the ring has a total resistance R, what is the magnitude of the current induced in the loop? Make a sketch of induced current vs time.

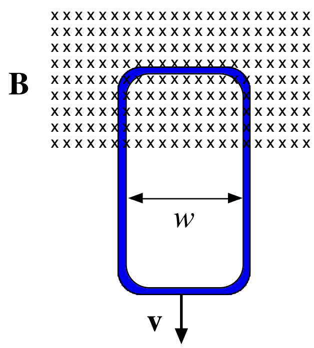
iv) [2 pts] In what direction does the current induced in the loop flow? clockwise or counter-clockwise around the $\hat{z}$ axis? (With the $\hat{z}$ axis pointing out of the page, draw the direction of the current.)

v) [3 pts] What is the total charge Q that is transported around the loop? Show that it does not matter exactly how the current decays from I to $I/2$, the transported charge Q is always the same!

vi) [3 pts] Suppose, as is actually more realistic, that the wire loop also has a self inductance L. Now sketch how the induced current varies as a function of time. Label on your sketch all relevant time scales. You may assume that the time interval Δt is very long (i.e. longer than any other time scale in the problem).

(part b on next page)

b) A long rectangular conducting loop of width w and length L is falling vertically through a uniform magnetic field $\mathbf{B}$ that is oriented perpendicular to the plane of the loop. Assume that as the loop falls, the upper portion of the loop remains in the region of magnetic field while the lower portion of the loop is outside the region of the magnetic field, as shown in the diagram below. $\mathbf{B}$ points into the page. The total mass of the loop is M and the loop has a total resistance R.

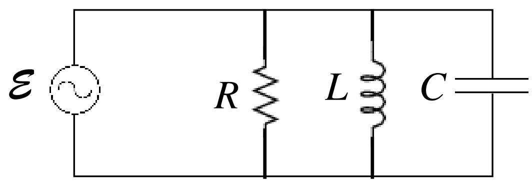


i) [6 pts] As the loop falls, it will reach a constant *terminal velocity* $\mathbf{v}$. What is the terminal velocity of the loop?

ii) [3 pts] Suppose the loop also has a self inductance L. Now what is the terminal velocity of the loop?

Problem 4 [25 points total]

Consider the circuit below, with a capacitor C, an inductor L, and a resistor R, all in parallel with an *ac* voltage source $\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$.



a) [2 pts] What is the current $I_R(t)$ flowing through the resistor?

b) [3 pts] What is the current $I_L(t)$ flowing through the inductor?

c) [3 pts] What is the current $I_C(t)$ flowing through the capacitor?

d) [3 pts] What is the total current $I(t) = I_R(t) + I_L(t) + I_R(t)$ flowing out of the voltage source? What is the amplitude of $I(t)$ and what is its phase φ relative to the voltage source $\mathcal{E}(t)$?

e) [3 pts] Sketch the amplitude of $I(t)$. At what frequency ω is $I(t)$ a minimum? Give a brief physical explanation of what is happening at this current minimizing frequency.

f) [3 pts] At what frequency ω, if any, will the current $I(t)$ be in phase ($\varphi = 0$) with the voltage source $\mathcal{E}(t)$? What is the phase φ as $\omega \to 0$? What is the phase φ as $\omega \to \infty$?

g) [3 pts] What is the total time averaged power P dissipated in the circuit? How does it depend on frequency ω?

h) [5 pts] Suppose at $t = 0$, the voltage supply is removed (so that the left most branch of the circuit becomes an open circuit). Find a set of differential equation that determine how the currents I_L, I_R and I_C will behave in time. The currents will oscillate and decay to zero. What is the time constant of their decay? (Hint: try to eliminate variables in the system of differential equations until you get a 2nd order differential equation that involves only one of the currents alone.)

Problem 5 [20 points total]

a) Consider an electromagnetic wave in the vacuum whose electric field is given by

$$\mathbf{E}_1(\mathbf{r}, t) = E_0 \hat{x} \cos(kz - \omega t)$$

i) [4 pts] If the frequency of the wave is $f = 10^{15} Hz$. What is the wavelength of the wave (give a numerical value)? What is the value of ω? What is the value of k?

ii) [4 pts] Write an expression for the magnetic field component of the wave. Make sure to specify its direction. What is the amplitude of the magnetic field oscillation?

iii) [6 pts] A second electromagnetic wave has its electric field component given by

$$\mathbf{E}_2(\mathbf{r}, t) = E_0 \hat{y} \sin(kz - \omega t)$$

with the same values of E_0, k and ω as the first wave. Consider the superposition

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

What is the amplitude $|\mathbf{E}(\mathbf{r}, t)|$? If an observer sits at the origin $\mathbf{r} = 0$, describe how the electric field $\mathbf{E}(0, t)$ behaves as a function of time t.

b) [6 points] Consider electromagnetic fields given by

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= E_0 \hat{z} \cos kx \cos ky \cos \omega t \\
\mathbf{B}(\mathbf{r}, t) &= B_0 (\hat{x} \cos kx \sin ky - \hat{y} \sin kx \cos ky) \sin \omega t
\end{aligned}$$

Find the relation between E_0 and B_0 and between ω and k, if the above fields are to be a solution of Maxwell's equations in a vacuum.