Energy stored in an inductor

Turn current on from zero to I in an inductor. To drive the current one must do work against the induced emf of the inductor

$$dW = \mathcal{E} I \, dt = L I \frac{dI}{dt} dt = \frac{1}{2} L \frac{d(I^2)}{dt} dt$$

$$W = \int dW = \int dt \, \frac{1}{2} L \frac{d(I^2)}{dt} = \frac{1}{2} L I^2$$

energy stored in inductor with constant current I flowing is

$$U_L = \frac{1}{2} L I^2$$

(compare to energy stored in capacitor with constant voltage across it. $U_C = \frac{1}{2} C V^2$)

Consider now a long solenoid of length l and cross sectional area A. We previously computed its self inductance

$$L = \frac{4\pi}{c^2} A N^2 l \qquad N = \#\,\text{turns}/\text{length}$$

So $$U_L = \frac{1}{2} \frac{4\pi}{c^2} A l N^2 I^2$$

$$= \frac{1}{8\pi} \left(\frac{4\pi}{c} N I\right)^2 A l = \frac{1}{8\pi} B^2 (\text{Vol})$$

This is an example of a more general result which we will not prove:

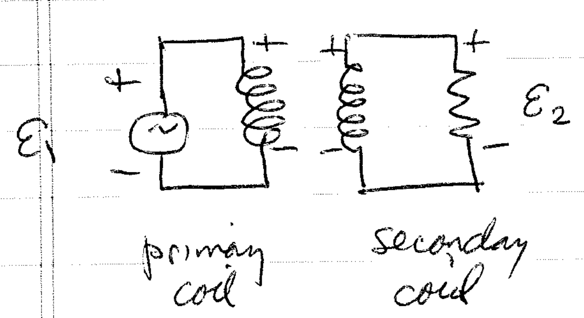
The energy stored in a magnetic field in some volume of space V is

$$U_B = \frac{1}{8\pi} \int_V dV \; B^2$$

compare to energy stored in an electric field

$$U_E = \frac{1}{8\pi} \int_V dV \; E^2$$

Transformers



inducting coils configured so that same magnetic flux penetrates each loop of both coils

$\mathcal{E}_1$ is time varying voltage supply (usually ac source)
$\mathcal{E}_2$ is voltage drop across load

primary: $\mathcal{E}_1 - \frac{1}{c} N_1 \frac{d\Phi_1}{dt} = 0 \Rightarrow \mathcal{E}_1 = \frac{1}{c} N_1 \frac{d\Phi_1}{dt}$

Φ_1 is flux through one turn of the coil, N_1 is total number of turns

secondary: $\mathcal{E}_2 - \frac{1}{c} N_2 \frac{d\Phi_2}{dt} = 0 \Rightarrow \mathcal{E}_2 = \frac{1}{c} N_2 \frac{d\Phi_2}{dt}$

Φ_2 is flux through one turn of the coil, N_2 is total number of turns

By construction $\Phi_1 = \Phi_2$ so

$$\boxed{\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}}$$

if $N_2 > N_1$ then $\mathcal{E}_2 > \mathcal{E}_1$ "step up" transformer

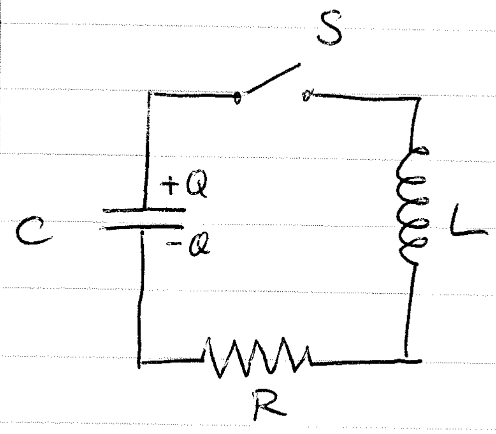
if $N_2 < N_1$ then $\mathcal{E}_2 < \mathcal{E}_1$ "step down" transformer

In an <u>ideal transformer</u> the power delivered to the secondary coil is equal to the power ~~lost~~ in primary coil.

$$\Rightarrow \quad I_1 \mathcal{E}_1 = I_2 \mathcal{E}_2$$

power dissipated in primary coil — power dissipated in load in secondary coil

$$\Rightarrow \quad \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{I_1}{I_2} \quad \text{so} \quad \boxed{\frac{I_2}{I_1} = \frac{N_1}{N_2}}$$



Let V be voltage drop across capacitor. $V = \frac{Q}{C}$

When switch S closes, then V will act like an emf driving current I clockwise around loop.

Kirchoff 2nd law

$$\Rightarrow V = L\frac{dI}{dt} + RI$$

Since I results from Q flowing from + plate of capacitor to − plate of capacitor, $I = -\frac{dQ}{dt}$
(as Q decreases, current flows)

From $V = \frac{Q}{C}$ we have $\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt} = -\frac{1}{C} I$

$$\frac{d^2V}{dt^2} = -\frac{1}{C}\frac{dI}{dt}$$

Substituting in above gives differential eqn for V

$$V = -LC\frac{d^2V}{dt^2} - RC\frac{dV}{dt}$$

or $$\boxed{\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC}V = 0}$$

just like damped harmonic oscillator

$$m\frac{d^2x}{dt} = -kx - \alpha\frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

ma

Hook's law force from spring

friction

Solution is a damped harmonic oscillation

Try

$$V(t) = Ae^{-\alpha t}\cos\omega t$$

$$\frac{dV}{dt} = -\alpha A e^{-\alpha t}\cos\omega t - \omega A e^{-\alpha t}\sin\omega t$$

$$\frac{d^2V}{dt^2} = +\alpha^2 A e^{-\alpha t}\cos\omega t + \alpha\omega A e^{-\alpha t}\sin\omega t$$
$$+ \alpha\omega e^{-\alpha t}\sin\omega t - \omega^2 A e^{-\alpha t}\cos\omega t$$

plug in

$$(\alpha^2-\omega^2)Ae^{-\alpha t}\cos\omega t + 2\alpha\omega Ae^{-\alpha t}\sin\omega t$$

$$-\frac{R}{L}\alpha A e^{-\alpha t}\cos\omega t - \frac{R}{L}\omega A e^{-\alpha t}\sin\omega t$$

$$+\frac{1}{LC}Ae^{-\alpha t}\cos\omega t = 0$$

$$\Rightarrow \left[\left(\alpha^2-\omega^2-\frac{R\alpha}{L}+\frac{1}{LC}\right)\cos\omega t + \left(2\alpha\omega - \frac{R}{L}\omega\right)\sin\omega t\right]Ae^{-\alpha t} = 0$$

if above $= 0$ at all times t, it can only be because coefficients of $\cos\omega t$ and $\sin\omega t$ terms vanish

$$\Rightarrow \begin{cases} (\alpha^2 - \omega^2 - \frac{R\alpha}{L} + \frac{1}{LC}) = 0 & ① \\ (2\alpha\omega - \frac{R}{L}\omega) = 0 \quad \Rightarrow \quad \boxed{\alpha = \frac{R}{2L}} & ② \end{cases}$$

sub into ①

$$\frac{R^2}{4L^2} - \omega^2 - \frac{R^2}{2L^2} + \frac{1}{LC} = 0$$

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \qquad \boxed{\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

ω real $\Rightarrow \frac{1}{LC} > \frac{R^2}{4L^2} \Rightarrow$ weak damped circuit

Note, if $R=0$, then no dissipation: $\alpha = 0$, $\omega = \sqrt{\frac{1}{LC}}$
we have pure harmonic oscillation with freq ω

$$V(t) = V_0 \cos \omega t$$

~~Adding the resistor provide~~

$$I(t) = -C\frac{dV}{dt} = CV_0\omega \sin \omega t$$

energy is oscillating between capacitor and inductor
ie between electric field and magnetic field energy

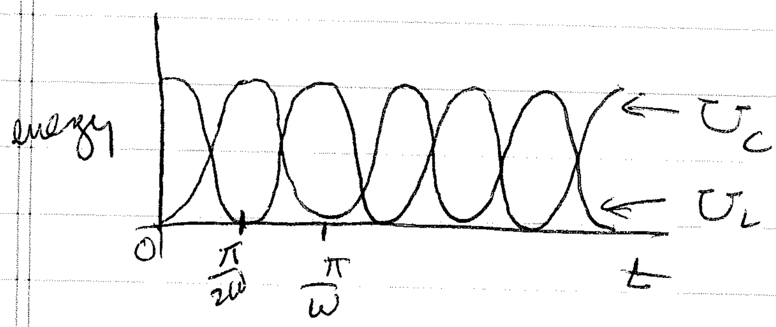
energy capacitor $U_C = \frac{1}{2}CV^2 = \frac{1}{2}CV_0^2\cos^2\omega t$

energy inductor $U_L = \frac{1}{2}LI^2 = \frac{1}{2}LC^2V_0^2\omega^2\sin^2\omega t$

using $\omega^2 = \frac{1}{LC}$, $U_L = \frac{1}{2}CV_0^2\sin^2\omega t$

$$U_L + U_C = \frac{1}{2}CV_0^2(\cos^2\omega t + \sin^2\omega t)$$
$$= \frac{1}{2}CV_0^2 \quad \text{constant in time}$$

Just like energy oscillates between kinetic and potential in the harmonic oscillator



Note: our solution had U_C max at $t=0$.
But we could always shift the time scale $t' = t + t_0$, so $t = t' - t_0$, to get solution

$$V(t) = V_0 e^{-\alpha t'} e^{\alpha t_0} \cos(\omega t' - \omega t_0)$$

$$V(t) = V_0' e^{-\alpha t'} \cos(\omega t' - \delta)$$

$\delta = \omega t_0$
$V_0' = V_0 e^{+\alpha t_0}$

general solution can have the oscillation phase shifted by any amount δ.

$$\cos(\omega t - \delta) = \cos \omega t \cos \delta - \sin \omega t \sin \delta$$

or equivalently be any linear combination of cosine and sine.

Adding the resistor serves to dissipate energy (energy goes into heating up the resistor)

Now $V(t) = V_0 \cos\omega t \, e^{-\alpha t}$

$$I(t) = -C\frac{dV}{dt} = CV_0\omega \sin\omega t \, e^{-\alpha t} + \alpha C V_0 \cos\omega t \, e^{-\alpha t}$$

$$I(t) = CV_0\omega \left[\sin\omega t + \frac{\alpha}{\omega}\cos\omega t \right] e^{-\alpha t}$$

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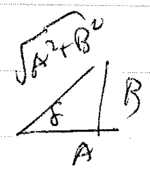
we can always write $A\sin\omega t + B\cos\omega t = C\sin(\omega t + \delta)$

$$= C\cos\delta \sin\omega t + C\sin\delta\cos\omega t$$



$$\Rightarrow \begin{cases} A = C\cos\delta \\ B = C\sin\delta \end{cases} \Rightarrow \frac{B}{A} = \tan\delta$$

$$\Rightarrow \cos\delta = \frac{A}{\sqrt{A^2+B^2}} \quad \text{and} \quad C = \frac{A}{\left(\frac{A}{\sqrt{A^2+B^2}}\right)} = \sqrt{A^2+B^2}$$

So $A\sin\omega t + B\cos\omega t = \sqrt{A^2+B^2}\,\sin(\omega t + \delta)$ $\quad \delta = \arctan\frac{B}{A}$

$$I(t) = CV_0\omega\sqrt{1+\left(\frac{\alpha}{\omega}\right)^2}\,\sin(\omega t+\delta)\,e^{-\alpha t} \qquad \tan\delta = \frac{\alpha}{\omega}$$

$$= CV_0\sqrt{\omega^2+\alpha^2}\,\sin(\omega t+\delta)\,e^{-\alpha t}$$

$$= CV_0\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2} + \frac{R^2}{4L^2}}\,\sin(\omega t+\delta)\,e^{-\alpha t}$$

$$= \frac{CV_0}{\sqrt{LC}}\,\sin(\omega t+\delta)\,e^{-\alpha t}$$

$$U_C = \tfrac{1}{2} C V_0^2 \cos^2 \omega t \; e^{-2\alpha t}$$

$$U_L = \tfrac{1}{2} L I^2 = \tfrac{1}{2} L \frac{C^2 V_0^2}{LC} \sin^2(\omega t + \delta) e^{-2\alpha t}$$

$$= \tfrac{1}{2} C V_0^2 \sin^2(\omega t + \delta) e^{-2\alpha t}$$

instead of being $\frac{\pi}{2}$ out of phase, U_C and U_L are now $\frac{\pi}{2} - \delta$ out of phase

Quality factor

$$Q \equiv 2\pi \frac{\text{energy stored at start of one cycle}}{\text{energy lost in one cycle}}$$

The smaller the dissipation from R, the larger Q

energy stored at start of one cycle is $\frac{1}{2} C V_0^2$

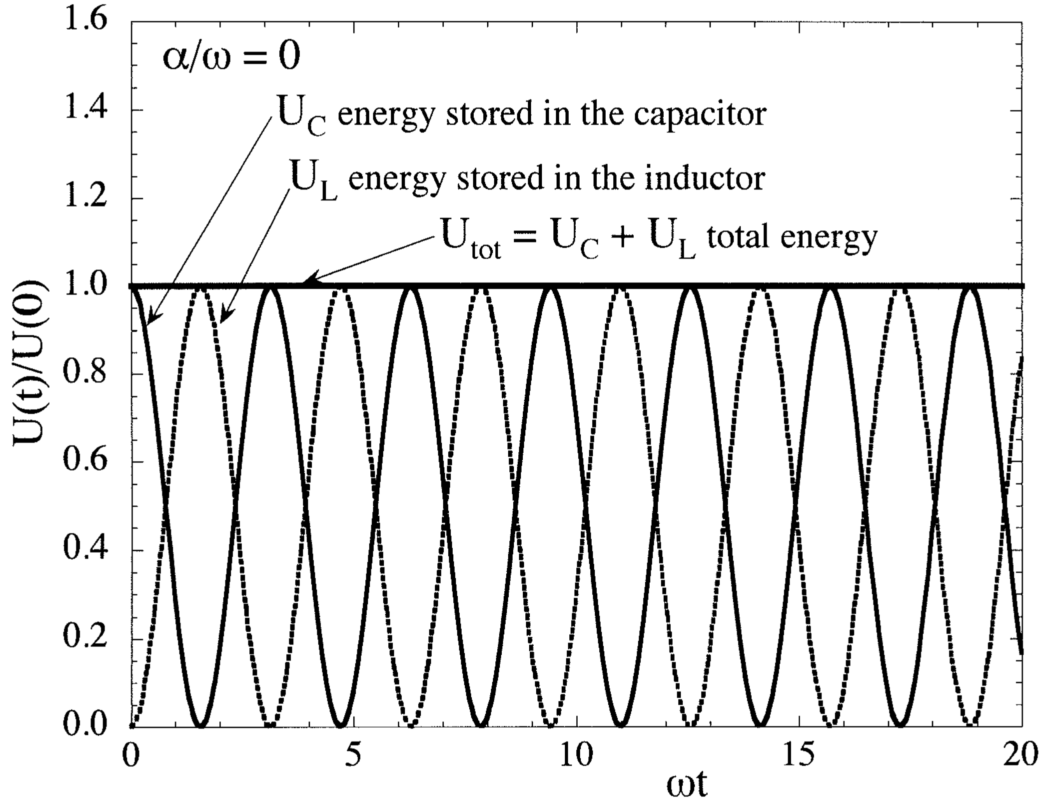
energy stored at end of one cycle is $\frac{1}{2} C V_0^2 e^{-2\alpha T}$

where $T = \frac{2\pi}{\omega}$ is period of oscillation

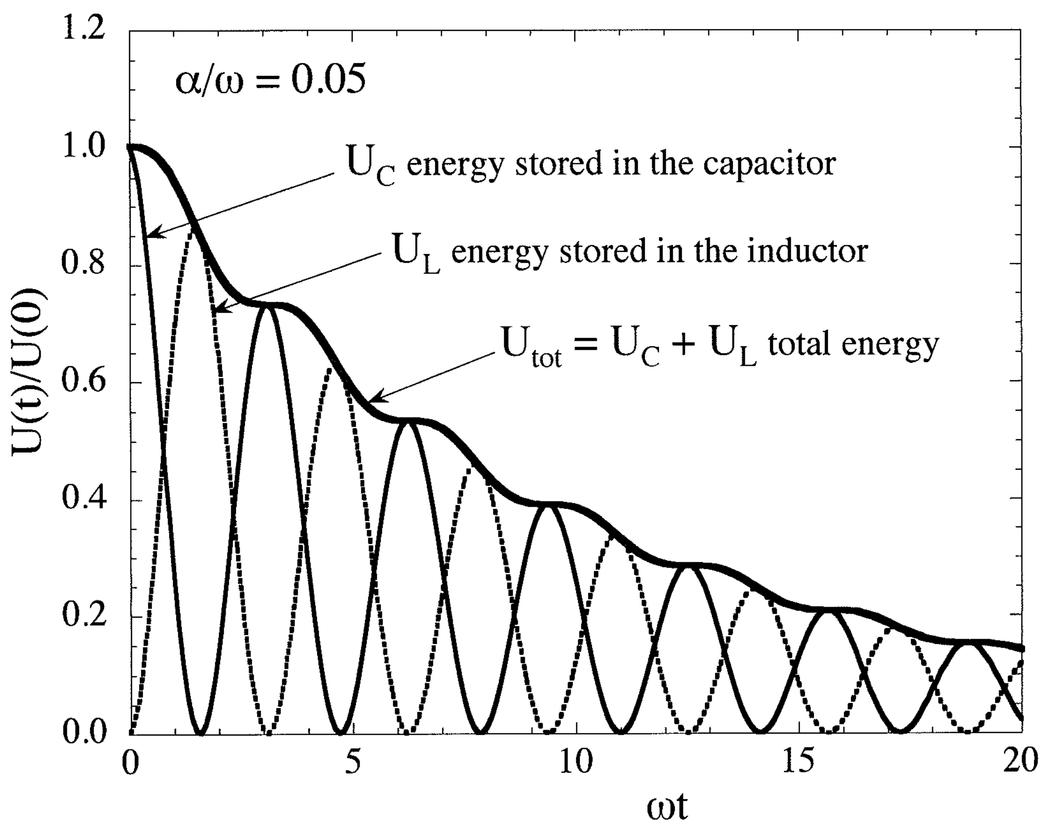
$$Q = 2\pi \frac{\frac{1}{2} C V_0^2}{\frac{1}{2} C V_0^2 (1 - e^{-2\alpha T})} = \frac{2\pi}{(1 - e^{-2\alpha T})}$$

For weak damping $\alpha T = \frac{\alpha 2\pi}{\omega} \ll 1$

$$Q \simeq \frac{2\pi}{1 - (1 - 2\alpha T)} = \frac{2\pi}{2\alpha T} = \frac{2\pi \, \omega}{2 \alpha 2\pi} = \frac{\omega}{2\alpha}$$



α/ω = 0
U_C energy stored in the capacitor
U_L energy stored in the inductor
U_tot = U_C + U_L total energy
U(t)/U(0)
ωt



α/ω = 0.05
U_C energy stored in the capacitor
U_L energy stored in the inductor
U_tot = U_C + U_L total energy
U(t)/U(0)
ωt

$$Q = \frac{\omega}{2\alpha} = \frac{\omega}{\left(\frac{2R}{2L}\right)} = \frac{L}{R}\,\omega$$

recall $\frac{L}{R}$ is decay time of L-R circuit

Overdamped oscillation

What if $\frac{1}{LC} < \frac{R^2}{4L^2}$ so that ω is imaginary?

Solution is $V(t) = Ae^{-\beta_1 t} + Be^{-\beta_2 t}$

$$\frac{dV}{dt} = -\beta_1 A e^{-\beta_1 t} - \beta_2 B e^{-\beta_2 t}$$

$$\frac{d^2V}{dt^2} = \beta_1^2 A e^{-\beta_1 t} + \beta_2^2 B e^{\beta_2 t}$$

$$\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC}V = 0$$

$$\Rightarrow \left(\beta_1^2 - \beta_1 \frac{R}{L} + \frac{1}{LC}\right)Ae^{-\beta_1 t} + \left(\beta_2^2 - \beta_2 \frac{R}{L} + \frac{1}{LC}\right)Be^{-\beta_1 t} = 0$$

$$\Rightarrow \beta_{1,2} = \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

If $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$ then

$$V(t) = (A + Bt)\,e^{-\beta t}$$