Maxwell's equations so far

1) $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ $\qquad \oint_S d\vec{a}\cdot\vec{E} = 4\pi Q_{encl}$ Gauss

2) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\qquad \oint_C \vec{E}\cdot d\vec{s} = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}$ Faraday

3) $\vec{\nabla}\cdot\vec{B} = 0$ $\qquad \oint_S d\vec{a}\cdot\vec{B} = 0$ no magnetic monopoles

4 $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$ $\qquad \oint_C \vec{B}\cdot d\vec{s} = \frac{4\pi}{c} I_{encl}$ Ampere

But something is missing from Ampere's law.

When we allow charge density ρ to vary in time, then ~~current~~ charge conservation gives

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \neq 0$$

But Ampere's law would imply

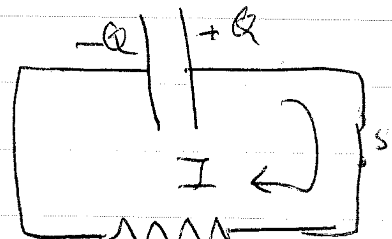
$$\vec{\nabla}\cdot\vec{J} = \frac{c}{4\pi}\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$$

but $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ for <u>any</u> vector function $\vec{B}$

Ampere's law implies that $\vec{\nabla}\cdot\vec{J} = 0$ always, <u>unless</u> Ampere's law is no longer correct when we have time varying charge densities!

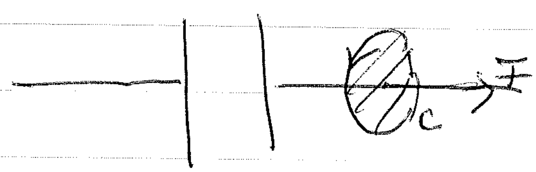
$\Rightarrow$ there is a term missing from Ampere's equation.

Another way to see it: Consider a capacitor discharging in an RC circuit



at $t=0$ capacitor C was charged up to charge Q, voltage $\frac{Q}{C}$. Then switch S is closed and capacitor charges carry current $I(t) = \frac{Q}{RC} e^{-t/RC}$ to flow around loop

Apply Ampere's law $\oint_C \vec{B}\cdot d\vec{s} = \frac{4\pi}{c} I_{encl}$ to the loop C shown below that circulates around the wire

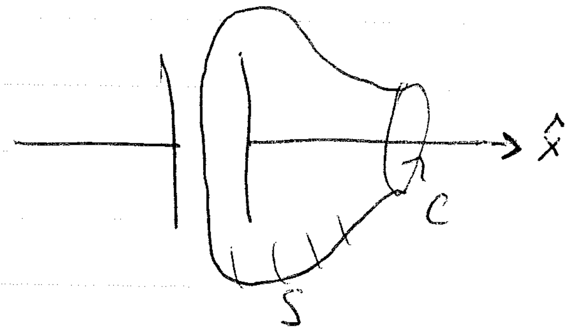


If one computes I_{encl} as the flux of current through the flat circle ~~enclosed~~ bounded by loop C then it seems like $I_{encl} \neq 0$ and there is a $\vec{B}$ circulating around the wire.

But we are free to take the surface S in

$$I_{encl} = \int_S d\vec{a}\cdot\vec{J}$$

to be any surface bounded by the loop C. So what if we took a surface that went between the plates of the capacitor and was not pierced by the wire carrying the current I.



Now $\int_S d\vec{a}\cdot\vec{J} = 0$ since no current flows between the capacitor plates

We would seem to have a contradiction - yet more ~~evidence~~ ~~that~~ Ampere's law must be missing a term.

It was Maxwell who realized the missing term. One way to see what it is is as follows.

We want the integral over the surface S above to give $\frac{4\pi}{c} I$. But there is no $\vec{J}$ passing through the surface. But there is an electric flux through the surface.

Imagine we can approx the capacitor as a pair of planes large compared to their separation so we can ignore edge effects. Then there is ~~a const~~ uniform electric field between the plates given by

$$\vec{E} = 4\pi \frac{Q}{A}\hat{n} \qquad \hat{n} \text{ points to left} \quad \hat{n} = -\hat{x}$$

The surface S passes between the plates and the flux of $\vec{E}$ through the surface is

$d\vec{a} = da\,\hat{x}$

(−) since $\vec{E}$ ad $d\vec{a}$ are antiparallel

$$\int_S d\vec{a}\cdot\vec{E} = -4\pi Q$$

Q is charge on right hand plate

therefore $\frac{1}{c}\frac{\partial}{\partial t}\left[\oint_{S'} d\vec{a}\cdot\vec{E}\right] = -\frac{4\pi}{c}\frac{dQ}{dt} = \frac{4\pi}{c} I$

since $I = -\frac{dQ}{dt}$

so $\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ plays the role of a current density

that will integrate to give the needed I_{encl}

So Ampere's law should be

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Maxwell called $\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$ the "displacement current"

The displacent current also fixes up the problem Ampere's law had with charge conservation. Now.

$$0 = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J} + \frac{1}{c}\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} = -\frac{1}{4\pi}\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} = -\frac{1}{4\pi}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E})$$

$$= -\frac{1}{4\pi}\frac{\partial}{\partial t}(4\pi\rho) = -\frac{\partial\rho}{\partial t}$$

and we regain current conservation!

Electromagnetic Waves

Source free Maxwell's Equations $(\rho = 0, \vec{J} = 0)$
no currents or charges — in a vacuum

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Now Maxwell's Equations look very symmetric between E and B.

We want to look for a solution to above source free equations. Can there be electric and magnetic fields even when there are no charges or currents? YES!

Consider a solution in which $\vec{B}$ points only in the $\hat{x}$ direction: $\vec{B}(\vec{r},t) = B(\vec{r},t)\,\hat{x}$

Faradays law then becomes

$$-\frac{1}{c}\frac{\partial B}{\partial t} = (\vec{\nabla}\times\vec{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}$$

we see that $B\hat{x}$ has an effect on E_z and E_y
Let us ~~assume~~ look for a solution where $\vec{E}$ points only in the z direction. Then $\vec{E} = E_z\,\hat{z}$ and,

$$-\frac{1}{c}\frac{\partial B}{\partial t} = \frac{\partial E}{\partial y}$$

Ampere's law is then

$$\frac{1}{c}\frac{\partial E}{\partial t} = (\vec{\nabla}\times\vec{B})_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{\partial B}{\partial y} \qquad \text{since} \begin{cases} B_x = B \\ B_y = 0 \end{cases}$$

$$\frac{1}{c}\frac{\partial E}{\partial t} = -\frac{\partial B}{\partial y} \qquad \text{Ampere}$$

$$\frac{\partial E}{\partial y} = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \text{Faraday}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\text{Ampere}) \Rightarrow \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = -\frac{1}{c}\frac{\partial^2 B}{\partial t\,\partial y}$$

$$\frac{\partial}{\partial y}(\text{Faraday}) \Rightarrow \frac{\partial^2 E}{\partial y^2} = -\frac{1}{c}\frac{\partial^2 B}{\partial y\,\partial t} \qquad \text{but } \frac{\partial^2 B}{\partial t\,\partial y} = \frac{\partial^2 B}{\partial y\,\partial t}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial y^2}$$

$$\text{or} \quad \frac{\partial^2 E}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

This is the <u>wave equation</u>!
speed of wave is c

~~Also: Gauss law~~ $\cancel{\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \frac{\partial E}{\partial z} = 0 \Rightarrow E \text{ indep of } z}$

$\cancel{\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \frac{\partial B}{\partial x} = 0 \Rightarrow B \text{ indep of } x}$

Similarly:

$$\frac{\partial}{\partial y}(\text{Ampere}) \Rightarrow \frac{1}{c}\frac{\partial E}{\partial y\,\partial t} = -\frac{\partial^2 B}{\partial y^2}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\text{Faraday}) \Rightarrow \frac{1}{c}\frac{\partial E}{\partial t\,\partial y} = -\frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}$$

$$\rightarrow \frac{\partial^2 B}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0$$

B also must satisfy the wave equation!

Also: Gauss Law $\vec{\nabla}\cdot\vec{E}=0 \Rightarrow \frac{\partial E}{\partial z}=0 \Rightarrow E$ indep of z

$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \frac{\partial B}{\partial x}=0 \Rightarrow B$ indep of x

Look for solutions where E and B depend only on y and time t

$$\frac{\partial^2 E(y,t)}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 E(y,t)}{\partial t^2} = 0$$

$$\frac{\partial^2 B(y,t)}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 B(y,t)}{\partial t^2} = 0$$

if E depended ~~also~~ also on x then by Faraday's law

$$-\frac{\partial E_z}{\partial x} = -\frac{1}{c}\frac{\partial B_y}{\partial t}$$

we would create a B_y. Since we want B only along $\hat{x}$, Therefore E cannot depend on x

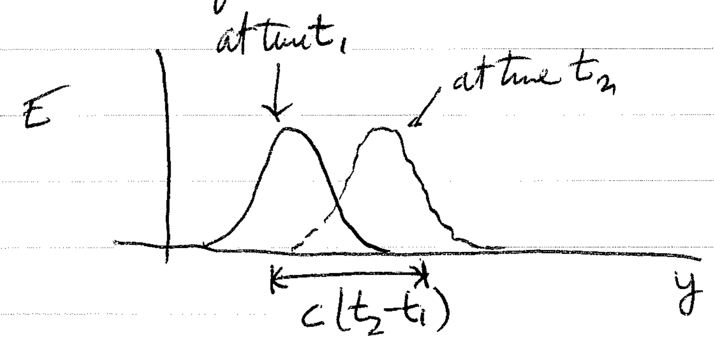
Consider any function $E(y,t)$ where coordinates y and t enter only in the combination $y-ct$, ie $E(y-ct)$ ~~Then~~ is a function only of the single variable $u \equiv y-ct$

Then $\frac{\partial E}{\partial y} = \frac{\partial E}{\partial u}$, $\frac{\partial^2 E}{\partial y^2} = \frac{\partial^2 E}{\partial u^2}$

$\frac{\partial E}{\partial t} = -c\frac{\partial E}{\partial u}$, $\frac{\partial^2 E}{\partial t^2} = c^2\frac{\partial^2 E}{\partial u^2}$

$$\frac{\partial^2 E}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial u^2} - \frac{1}{c^2}c^2\frac{\partial^2 E}{\partial u^2} = 0 \ !$$

So any function $E(y-ct)$ will solve the wave equation!



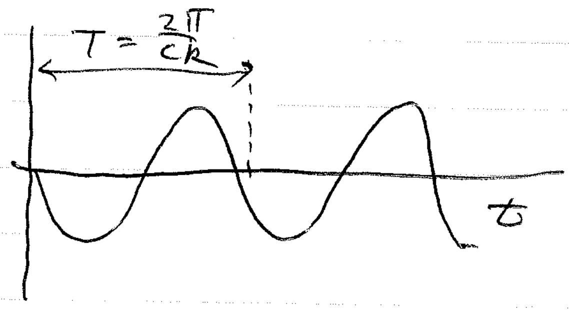
$E(u)$ has same shape at all times, but travels in $+y$ direction with speed c

Particularly important solutions are those that oscillate in time with a single frequency. We can get such a solution by taking:

$$E(y,t) = E_0 \sin[k(y-ct)]$$
$$= E_0 \sin(ky - ckt)$$

At constant position y in space, $E(y,t)$ looks like

at $y=0$ for example



period of oscillation is

$$T = \frac{2\pi}{ck}$$

$\Rightarrow$ frequency of oscillation is

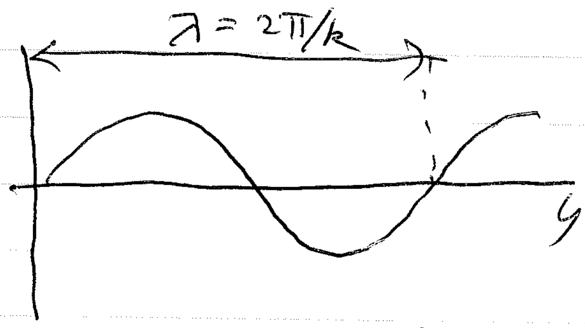
$$f = \frac{1}{T} = \frac{ck}{2\pi}$$

$\Rightarrow$ angular freq of oscillation is

$$\boxed{\omega = 2\pi f = ck}$$

At constant time, E varies in space like

at $t=0$ for example



$E(y)$ repeats itself with <u>wavelength</u>

$$\lambda = \frac{2\pi}{k}$$

$k = \frac{2\pi}{\lambda}$ is called the <u>wave number</u>

$$\vec{E}(\vec{r},t) = \hat{z} E_0 \sin(kx - \omega t)$$