This is solution for the electric field part of the wave. The corresponding magnetic field part is obtained by

Faraday: $\frac{\partial B}{\partial t} = -c \frac{\partial E}{\partial y} = -ckE_0 \cos(ky - \omega t)$
$= -\omega E_0 \cos(ky - \omega t)$

integrate to get

$$B(t) = +E_0 \sin(ky - \omega t)$$
$$= B_0 \sin(ky - \omega t) \qquad B_0 = E_0$$

<u>EM plane wave</u>

$$\left.\begin{aligned} \vec{E}(\vec{r},t) &= \hat{z}\, E_0 \sin(ky - \omega t) \\ \vec{B}(\vec{r},t) &= \hat{x}\, B_0 \sin(ky - \omega t) \end{aligned}\right\} \quad \text{with } \omega = ck, \quad B_0 = E_0$$

It is called a "plane" wave since $\vec{E}$ = constant, $\vec{B}$ = constant on any xz plane at fixed y, t

Amplitudes vary only in direction of propagation of wave, and not within the plane perpendicular to that direction

1) $\vec{E}$ and $\vec{B}$ oscillate in phase, ie $\vec{B}$ is a max when E is a max
2) $\vec{E} + \vec{B} \perp$ direction of propagation
$\vec{E} \times \vec{B}$ gives direction of propagation
($\vec{E}, \vec{B}$, direc of prop) form a right handed coord system

Maxwell's Eqn in MKS.

CGS:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

MKS:

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = \vec{0}$$
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Displacement current in MKS is $\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ so that

$$0 = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\vec{\nabla}\cdot\vec{J} + \mu_0\varepsilon_0\,\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t}$$
$$= \mu_0\vec{\nabla}\cdot\vec{J} + \mu_0\frac{\partial}{\partial t}(\varepsilon_0\vec{\nabla}\cdot\vec{E})$$
$$= \mu_0\vec{\nabla}\cdot\vec{J} + \mu_0\frac{\partial\rho}{\partial t} = 0 \text{ by } \cancel{\text{current}}\text{ charge conservation}$$

If one repeated the above steps for the EM wave, ~~it goes like~~ again for $\vec{E} = E(y)\hat{z}$ and $\vec{B} = B(y)\hat{x}$, we get

Faraday $$-\frac{\partial B}{\partial t} = \frac{\partial E}{\partial y} \Rightarrow -\frac{\partial^2 B}{\partial y\,\partial t} = \frac{\partial^2 E}{\partial y^2}$$

Ampere $$-\frac{\partial B}{\partial y} = \mu_0\varepsilon_0\frac{\partial E}{\partial t} \Rightarrow -\frac{\partial^2 B}{\partial t\,\partial y} = \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 E}{\partial y^2} - \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} = 0$$

wave equation in which speed of wave is $v = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$

When Maxwell derived his equations, and his solution for waves, he used MKS units with ε_0 and μ_0 as determined from purely electric and magnetic experiments. He was surprised to find that the numerical value of the velocity $\frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ was exactly the same as the velocity of light as measured by optical techniques.

This led Maxwell to the conclusion that light was in fact electromagnetic waves.

"The velocity of transverse undulations in our hypothetical medium, calculated from the electro-magnetic experiments of M.M. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena."

— 1862 Maxwell

Source free Maxwell equations are linear

If $\vec{E}_1$ and $\vec{B}_1$ and $\vec{E}_2$ and $\vec{B}_2$ are both solutions then so are $\vec{E}_1 + \vec{E}_2$ and $\vec{B}_1 + \vec{B}_2$

Consider $\vec{E}_i = \hat{z}\, E_{i0} \sin(k_i y - \omega_i t)$ $\quad \omega_i = c k_i$

one could make a solution by adding a linear superposition of many such $\vec{E}_i$

$$\vec{E} = \hat{z} \sum_i E_{i0} \sin(k_i y - \omega_i t)$$

Consider at $t=0$

$$E(y,0) = \hat{z} \sum_i E_{i0} \sin(k_i y)$$

We know ~~that~~ from theory of Fourier series that we can add a linear superposition of different sine waves to create any function we like.
At later $t>0$ that function will just travel to right or left with speed c, since it is a solution of the wave equation. This is how our general solution to wave equation relates to the harmonic oscillator solutions

If you have not yet seen Fourier series, you will see them eventually in math + later in physics.

The source free Maxwell's Equations are linear

If $\vec{E}_1, \vec{B}_1$ and $\vec{E}_2, \vec{B}_2$ are both solutions, then

$$\vec{E}_1 + \vec{E}_2 \,,\; \vec{B}_1 + \vec{B}_2 \quad \text{is also a solution}$$

Suppose right traveling wave $\begin{cases} \vec{E}_1 = \hat{z}\, E_0 \sin(ky - \omega t) \\ \vec{B}_1 = \hat{x}\, B_0 \sin(ky - \omega t) \end{cases}$ $\quad \omega = ck \quad B_0 = E_0$

$\frac{\partial B}{\partial t} = -c\frac{\partial E}{\partial y}$

left traveling wave $\begin{cases} \vec{E}_2 = \hat{z}\, E_0 \sin(ky + \omega t) \\ \vec{B}_2 = -\hat{x}\, B_0 \sin(ky + \omega t) \end{cases}$ $\quad \omega = ck \quad B_0 = E_0$

(−) sign here is important!

Then $\vec{E} = \vec{E}_1 + \vec{E}_2 = \hat{z} E_0 \left[\sin(ky - \omega t) + \sin(ky + \omega t)\right]$

$$= \hat{z} E_0 \left[\sin ky \cos \omega t - \cos ky \sin \omega t + \sin ky \cos \omega t + \cos ky \sin \omega t\right]$$

$$\vec{E} = \hat{z}\, 2E_0 \sin ky \cos \omega t$$

and $\quad \vec{B} = -\hat{x}\, 2B_0 \cos ky \sin \omega t$

$$\left\{ \begin{array}{l} \vec{E} = 2E_0 \hat{z} (\sin ky)(\cos \omega t) \\ \vec{B} = -2B_0 \hat{x} (\cos ky)(\sin \omega t) \end{array} \right\} \quad \underline{\text{standing wave}}$$

$$\begin{cases} \vec{E} = \hat{z}\,2E_0 (\sin ky)(\cos \omega t) \\ \vec{B} = -\hat{x}\,2B_0 (\cos ky)(\sin \omega t) \end{cases} \qquad \omega = ck \ , \ B_0 = E_0$$

<u>Standing wave</u> superposition of wave to right with velocity $c\hat{y}$ and wave to left with velocity $-c\hat{y}$

In contrast to single plane wave, in the standing wave

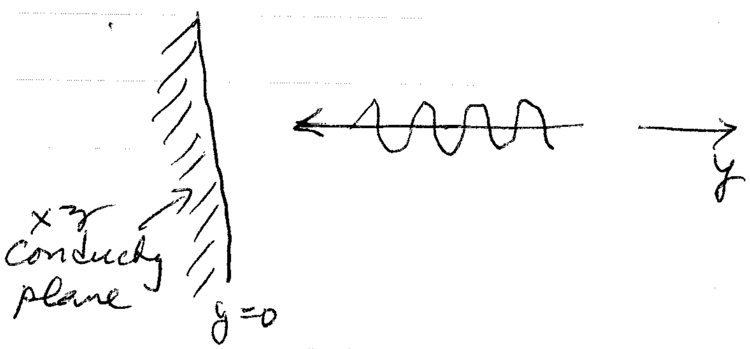
1) max of $|\vec{E}|$ occurs always at the same points in space – it does not travel with speed c similarly for max of $|\vec{B}|$
2) $\vec{E}$ and $\vec{B}$ oscillate $\frac{\pi}{2}$ out of phase: At $t=0$, $|\vec{E}|$ has its max value while $\vec{B}=0$ everywhere. At $t = \frac{T}{2} = \frac{\pi}{\omega}$, $\vec{E}=0$ everywhere and $|\vec{B}|$ has its max value
3) the max values of $|\vec{E}|$ and $|\vec{B}|$ are at different points in space, separated by one half wavelength $\frac{\lambda}{2} = \frac{\pi}{k}$

~~the standing wave has $\vec{E}=0$ at $y=0$ for all time t. the standing wave can therefore describe reflection of a wave from an infinite conducting plane at $y=0$. (since at surface of a conductor $E_{\parallel}=0$, the component of $\vec{E}$ $\parallel$ surface must vanish)~~

Since $\vec{B} = -\hat{x}\,2B_0 \sin \omega t$ at $y=0$

Reflection from a conducting surface

Suppose an infinite flat conducting plane fills the xz plane at $y=0$. An EM wave travels towards it from the ~~right~~ right side, hitting the plane. What is the solution for the resulting incident + reflected wave?



we have at the surface of the conductor at $y=0$

$$\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma\,\hat{y}$$

$$\vec{B}_{above} - \vec{B}_{below} = \frac{4\pi}{c}\,\vec{K}\times\hat{y}$$

we assume that inside the conductor the fields vanish. Then so $B_{below} = E_{below} = 0$

$$\vec{E}_{above} = 4\pi\sigma\,\hat{y}$$

But since our incoming wave is a plane wave we know $\vec{E} \perp \hat{y}$ ($\hat{y}$ is direc of prop)

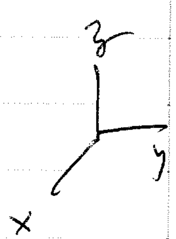
$$\Rightarrow \vec{E}_{above} = \vec{E}(y=0) = 0 \quad \text{all time}$$

$$\vec{B}_{above} = \frac{4\pi}{c} \vec{K} \times \hat{y}$$

A solution that serves $\vec{E}(y=0) = 0$ all time is just to add a reflected wave traveling to the right with equal amplitude as the incident wave. If we take $\vec{E}$ to be "polarized" in the $\hat{z}$ direction, with B in the $\hat{x}$ direction then the solution is the standing wave)

$$\begin{cases} \vec{E} = \hat{z}\, 2E_0 \sin(ky) \cos(\omega t) \\ \vec{B} = -\hat{x}\, 2E_0 \cos(ky) \sin(\omega t) \end{cases}$$

Then clearly $\vec{E}(y=0) = 0$ for all t



$$\vec{B}_{above} = \vec{B}(y=0) = -2E_0 \sin\omega t\ \hat{x} = \frac{4\pi}{c} \vec{K} \times \hat{y}$$

$\hat{z} \times \hat{y} = -\hat{x}$

$$\Rightarrow \vec{K} = \frac{cE_0}{2\pi} \sin\omega t\ \hat{z}$$

there is ac surface current density on the surface of the conductor that oscillates with freq ω of the incident wave

Energy in Electromagnetic Waves

Consider the plane wave

$$\vec{E}(r,t) = \hat{z}\, E_0 \sin(ky - \omega t)$$

$$\vec{B}(r,t) = \hat{x}\, B_0 \sin(ky - \omega t) \qquad \begin{array}{l} \omega = ck \\ B_0 = E_0 \end{array}$$

Electromagnetic fields contain energy.

The energy density (energy per unit volume) at a point $\vec{r}$ at time t is

$$\frac{1}{8\pi}|\vec{E}(\vec{r},t)|^2 + \frac{1}{8\pi}|\vec{B}(\vec{r},t)|^2$$

$$= \frac{1}{8\pi} E_0^2 \sin^2(ky - \omega t) + \frac{1}{8\pi} B_0^2 \sin^2(ky - \omega t)$$

$$= \frac{1}{4\pi} E_0^2 \sin^2(ky - \omega t) \qquad \text{since } B_0 = E_0$$

If we average over one ~~wavelength in space~~ period of oscillation in time, the average energy density becomes

$$\frac{1}{8\pi} E_0^2 \qquad \text{since average of } \sin^2(\theta) = \frac{1}{2}$$

In an EM wave, this ~~tag~~ energy propagates in the direction of the wave with speed c

Thus the average energy that passes, per unit time, through a unit area perpendicular to the direction of propagation is

$$S = \frac{E_0^2 c}{8\pi} \quad \frac{\text{energy}}{\text{time - vol}} = \text{power density}$$

or

$$S = \frac{\overline{E^2} c}{4\pi}$$

where $\overline{E^2}$ is the average of E^2 over one period of oscillation. For a sine wave $\overline{E^2} = \frac{1}{2} E_0^2$

Transmission of electromagnetic energy in the light waves from the sun hitting the earth is a major source of energy on Earth!

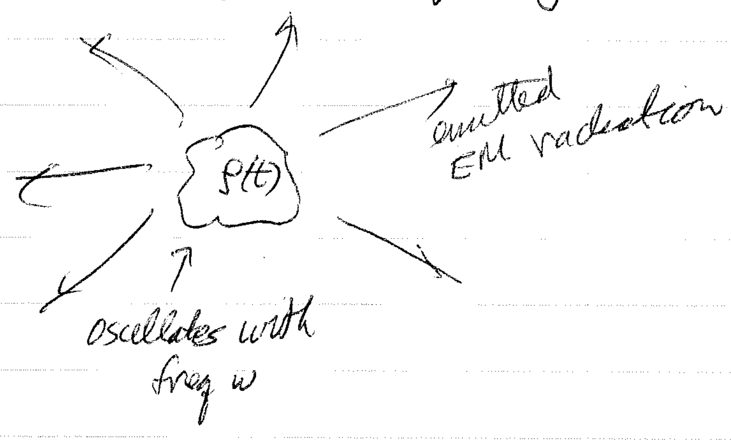
What causes EM waves?

charges at rest do NOT

charges in motion with constant velocity do NOT
(we can always transform to a frame of reference in which the charge is at rest)

It is accelerated charges that produce EM waves. The details of how will have to wait for a more advanced EM course. An attempt at a simple explanation can be found in Purcell Appendix B.

Suppose we have a charge distribution localized in space, that oscillates (and so is accelerating) with an angular frequency ω.

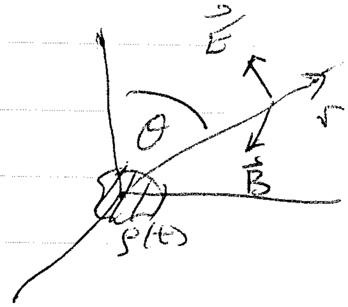


(radiation is another name for EM waves!)

The EM wave emitted by the oscillating charge does not go off as a collimated plane wave in a particular direction – it goes out radially in all directions.

electric field in such an outgoing spherical wave looks like

$$E(r,\theta,\varphi)\sin(kr-\omega t)\,\hat{\theta}$$



everywhere locally the wave looks like a plane wave going out in the radial direction

$\Rightarrow \vec{E}\perp\hat{r}$. For $\vec{E}\sim\hat{\theta}$ the $\vec{B}\sim\hat{\varphi}$

An important result is how the amplitude depends on distance r from the source. We must have

$E(r,\theta,\varphi)\sim\dfrac{f(\theta,\varphi)}{r}$ decrease as $\dfrac{1}{r}$

This is due to energy conservation. The electromagnetic power flowing through a sphere of radius r ~~cented~~ centered about the source should not depend on the value of the radius r, otherwise energy would be lost somewhere in going from r to r'.

Since power density $S\propto E^2$, and area of the sphere of radius r is $4\pi r^2$, we need

$$4\pi r^2 S \propto 4\pi r^2 E^2 = \text{constant}$$

$\Rightarrow E\propto\dfrac{1}{r}$ amplitude of E decreases as $\dfrac{1}{r}$ for spherical EM wave

Compare to E from a static point charge, where E decreases with distance from the charge as $\sim 1/r^2$.