on left side $\sigma_b = -|\vec{P}| = \vec{P}\cdot\hat{n}$ where $\hat{n}_{left} = -\hat{n}_{right}$

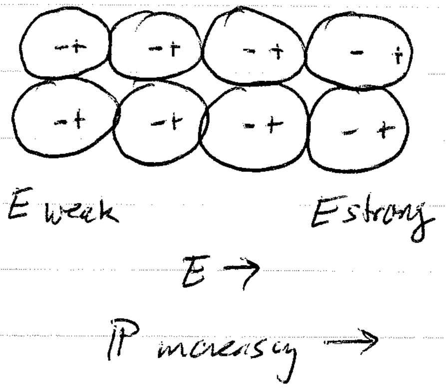
In general, a polarized dielectric will have surface charge density on its boundaries equal to

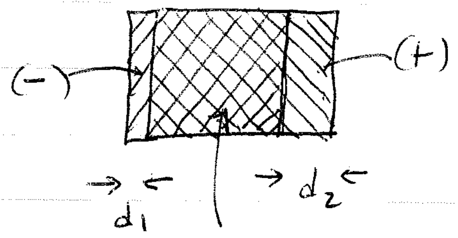
$$\boxed{\sigma_b = \vec{P}\cdot\hat{n}}$$

outward unit normal vector on surface

polarization density at surface

If the polarization density $\vec{P}$ is not uniform, there can also be bound charge <u>inside</u> the dielectric





net (-) region since (-) charge has been squeezed compared to (+) density

$d_2 > d_1$

Lets compute the net charge density inside the dielectric.

In practice, the lattice of + ions remains fixed and it is the electron clouds that displace. The + charge density inside remains as it was when $E = 0$. If total + charge is Q, length is L, and crosssectional area is A, then

$$\rho_+ = \frac{Q}{LA} \equiv \rho_0$$

The - charge density has been compressed to length L' since electrons move more on right side than left side since E is larger on the left

$$L' = L - (d_2 - d_1) = L - \Delta d \qquad \Delta d = d_2 - d_1$$

$$\rho_- = \frac{-Q}{AL'} = \frac{-Q}{AL(1-\frac{\Delta d}{L})}$$

assume $\frac{\Delta d}{L} \ll 1$ and expand

$$\rho_- \approx -\frac{Q}{AL}\left(1+\frac{\Delta d}{L}\right) = -\rho_0\left(1+\frac{\Delta d}{L}\right)$$

so <u>bound</u> charge density <u>inside</u> dielectric is

$$\rho_b = \rho_+ + \rho_- = \rho_0 - \rho_0\left(1+\frac{\Delta d}{L}\right) = -\rho_0\frac{\Delta d}{L}$$

Now $\rho_0 = nq$ $\qquad \rho = -nq\frac{\Delta d}{L}$

$\uparrow$ density of atoms

Polarization density is $\mathbb{P} = nqd$

So $\rho_b = -\frac{\Delta \mathbb{P}}{L} = -\frac{\partial \mathbb{P}}{\partial x}$ for a uniformly increasing $\vec{E}$

In general $\boxed{\rho_b = -\vec{\nabla}\cdot\vec{\mathbb{P}}}$

Since dielectric is neutral, the total bound charge should always vanish

total bound charge

$$\nearrow Q_b = \oint\limits_{\substack{S\\ \uparrow\\ \text{surface of}\\ \text{dielectric}}} da\, \sigma_b + \int\limits_{\substack{V\\ \uparrow\\ \text{volume of}\\ \text{dielectric}}} dV\, \rho_b$$

$$= \oint_S da\, \underbrace{\hat{n}\cdot\vec{P}}_{\sigma_b} + \int_V dV \left(\underbrace{-\vec{\nabla}\cdot\vec{P}}_{\rho_b}\right)$$

But by Gauss' theorem $\int_V dV\, \vec{\nabla}\cdot\vec{P} = \oint_S d\vec{a}\cdot\vec{P}$

$$Q_b = \oint_S \underbrace{d\vec{a}}_{da\,\hat{n}}\cdot\vec{P} - \oint_S d\vec{a}\cdot\vec{P} = 0$$ as needed!

So definitions of σ_b and ρ_b are consistent with neutrality of the dielectric

Bound charge due to polarized dielectric

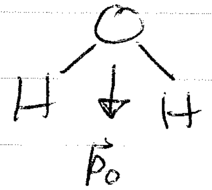
$$\boxed{\begin{array}{ll} \text{inside}: & \rho_b = -\vec{\nabla}\cdot\vec{P} \\ \text{on surface} & \sigma_b = \vec{P}\cdot\hat{n} \end{array}}$$

$\hat{n}$ is outward unit normal

These hold true no matter how the dielectric got polarized. We discussed so far one method

- a dielectric of neutral atoms (or molecules) where each atom has no electric dipole moment $\vec{p}$ when $\vec{E}=0$, but each atom has a ~~[illegible]~~ dipole moment induced
  $\vec{p} = \alpha\vec{E}$ in the presence of an electric field $\vec{E}$
  ↑ atomic polarizability

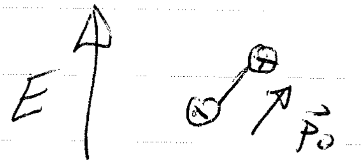
- but there are also molecules in which, even when $\vec{E}=0$, there is a net electric dipole moment $\vec{p}_0$. An example is water



There is net electric dipole pointing as shown.

Many other molecules have similar "built in" electric dipoles. However in most situations, in the dielectric liquid or gas, the molecules have random orientations and so at $\vec{E}=0$, the polarization density $\vec{P}=0$, since when we sum up many $\vec{p}_0$

each with random directions, the result averages to zero. When one turns ~~on~~ an electric field however, there is an interaction that tends to allign the electric dipole with the electric field



energy is lowest when $\vec{p}_0$ parallel to $\vec{E}$.

For a uniform electric field, $\vec{E} = -\vec{\nabla}\phi$
with $\phi(\vec{r}) = \phi_0 - \vec{E}\cdot\vec{r}$ $\quad$ $\vec{E}$ a constant vector
So energy of interaction of dipole with E is

$$U = \Big[q\,\phi(\underset{\substack{\uparrow \\ \text{position of} \\ + \text{ charge}}}{\vec{r}_+}) - q\,\phi(\underset{\substack{\nwarrow \\ \text{position of} \\ - \text{ charge}}}{\vec{r}_-})\Big]$$

$$= \Big[q\phi_0 - q\vec{E}\cdot\vec{r}_+ - q\phi_0 + q\vec{E}\cdot\vec{r}_-\Big]$$

$$= -q\vec{E}\cdot(\vec{r}_+ - \vec{r}_-) = -q\vec{E}\cdot\vec{d}$$

$$U = -\vec{E}\cdot\vec{p}_0 \qquad \vec{p}_0 = q\vec{d}$$

So U is lowest when $\vec{p}_0$ is parallel to $\vec{E}$. When the system is in thermal equilibrium at at temperature T, the ordering effect of the interaction potential energy above competes with the disordering effect of thermal fluctuations. Result is that ~~system develops a net polarization density~~ each molecule only ~~on average~~ partly alligns with $\vec{E}$, getting an average dipole moment

$$\vec{p} = \alpha \vec{E}$$

α increases as T decreases

This again leads to

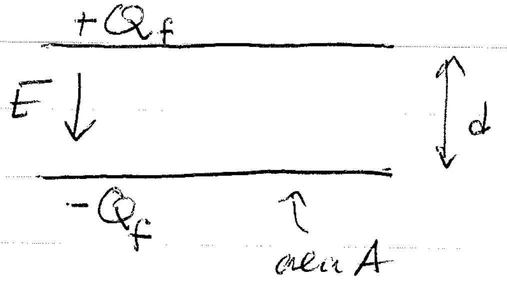
$$\vec{\mathbb{P}} = \chi_e \vec{E} \quad \text{with } \chi_e \simeq n\alpha$$

- But in some materials, where the basic unit component has a net electric dipole moment, and the units order in a regular array, the material can have a finite polarization density $\vec{\mathbb{P}}$ even when $\vec{E} = 0$. These materials are called *ferro electrics*.

In the below we will assume we are talking about a *linear* dielectric in which

$$\vec{\mathbb{P}} = \chi_e \vec{E}$$

Parallel plate Capacitor



suppose we have a parallel plate capacitor with "free" charge $+Q_f$ and $-Q_f$ on the plates.
By "free" charge we mean charge that the experimenter adds, as opposed to "bound" charge coming from polarization of a dielectric

If there is a vacuum in between the plates there will be an electric field between the plates

$$E = 4\pi\sigma_f \qquad \sigma_f = \frac{Q_f}{A} \quad \text{free surface charge}$$

potential drop across plates is $V = Ed$

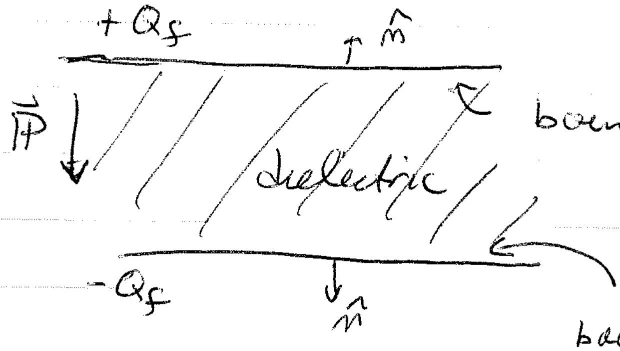
Capacitance of the plates is $C = \dfrac{Q_f}{V} = \dfrac{Q_f}{4\pi\left(\frac{Q_f}{A}\right)d} = \dfrac{A}{4\pi d}$

Now consider filling the space between the plates with a dielectric material with electric susceptibility χ_e

The electric field E between the plates polarizes the dielectric and gives

$$\vec{P} = \chi_e \vec{E}$$

Since $\vec{E}$ is uniform, then $\vec{P}$ is uniform. So there is no bound charge inside the dielectric ($\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$ since $\vec{P}$ is constant) but there is bound surface charge at the plates $\sigma_b = \vec{P}\cdot\hat{n}$



bound surface charge

$\sigma_b = -\chi_e E$ (− since $\vec{P}$ antiparallel to $\hat{n}$)

bound surface charge

$\sigma_b = +\chi_e E$ (+ since $\vec{P}$ parallel to $\hat{n}$)

Total surface charge on upper plate is then

$$\sigma_{tot} = \sigma_f \underbrace{- \chi_e E}_{\sigma_b}$$

total surface charge on bottom plate is

$$\sigma_f \equiv \frac{Q_f}{A}$$

$$-\sigma_{tot} = -\sigma_f \underbrace{+ \chi_e E}_{\sigma_b}$$

Electric field between plates is due to the total charge

$$E = 4\pi \sigma_{tot} = 4\pi \sigma_f - 4\pi \chi_e E$$

$$\Rightarrow \quad E(1 + 4\pi \chi_e) = 4\pi \sigma_f$$

$$E = 4\pi \left(\frac{\sigma_f}{1 + 4\pi \chi_e} \right)$$

$$\Rightarrow \quad \sigma_{tot} = \frac{\sigma_f}{1 + 4\pi \chi_e}$$

$$\sigma_b = \sigma_{tot} - \sigma_f = \frac{-4\pi \chi_e}{1 + 4\pi \chi_e} \sigma_f$$

σ_b has opposite sign as σ_f

One denotes

$$\boxed{\epsilon \equiv 1 + 4\pi \chi_e}$$ the dielectric constant

$$\sigma_{tot} = \frac{\sigma_f}{\epsilon}$$

If we denote $E_0 = 4\pi \sigma_f$ the electric field we would have between the plates if there was no dielectric there, then

$$E = \frac{E_0}{\epsilon}$$

Since $\chi_e > 0$, $\epsilon > 1$, and $\sigma_{tot} < \sigma$, $E < E_0$

Since E is reduced from E_0, and σ_{tot} is reduced from σ_f, due to the polarization of the dielectric, we say that the dielectric is "screening" the free charge (making it seem smaller).

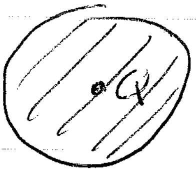
The voltage drop is now $V = Ed = \frac{E_0 d}{\epsilon} = \frac{V_0}{\epsilon}$
where V_0 is voltage drop without the dielectric

And so $C = \frac{Q_f}{V} = \frac{Q_f}{V_0}\epsilon = \epsilon \underset{\substack{\uparrow \\ \text{capacitance} \\ \text{w/o dielectric}}}{C_0}$

So inserting the dielectric increases the capacitance by a factor of ϵ

$$\boxed{C = \epsilon C_0}$$

Consider a sphere of dielectric material of radius R. A point charge Q is at the center of the sphere. What does the electric field look like? What is the bound charge?



Consider behavior inside the dielectric.
Gauss Law is

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_{tot} = 4\pi(\underset{\text{free charge}}{\rho_f} + \underset{\text{bound charge}}{\rho_b})$$

$$\rho_b = -\vec{\nabla}\cdot\vec{\mathbb{P}}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = 4\pi\rho_f - 4\pi\vec{\nabla}\cdot\vec{\mathbb{P}}$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E} + 4\pi\vec{\mathbb{P}}) = 4\pi\rho_f$$

Define displacement field $\vec{D} = \vec{E} + 4\pi\vec{\mathbb{P}}$

$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f$ can solve for $\vec{\mathbb{D}}$ in terms of ρ_f just like we used to solve for $\vec{E}$ in terms of ρ

For the sphere

$$\rightarrow \vec{D}(\vec{r}) = \frac{Q}{r^2}\hat{r}$$

$\vec{D}$ is what the electric field would be if there was only free charge

For a linear dielectric, $\vec{P} = \chi_e \vec{E}$

So $\vec{D} = \vec{E} + 4\pi \chi_e \vec{E} = (1 + 4\pi \chi_e)\vec{E} = \epsilon \vec{E}$

$$\boxed{\vec{D} = \epsilon \vec{E}} \quad \text{or} \quad \vec{E} = \vec{D}/\epsilon$$

So <u>inside the sphere</u> $\boxed{\vec{E} = \frac{Q}{\epsilon r^2}\hat{r}}$ $r < R$

the is electric field from a point charge at origin with total charge

$$Q_{tot} = \frac{Q}{\epsilon}$$

The dielectric screens the point charge Q to reduce it to a total charge Q/ϵ.

$$Q_{tot} = \frac{Q}{\epsilon} = Q + Q_b \Rightarrow Q_b = \left(\frac{1}{\epsilon} - 1\right)Q$$
$$= \left(\frac{1-\epsilon}{\epsilon}\right)Q$$

bound charge at origin is: $\boxed{Q_b = \frac{-4\pi \chi_e Q}{\epsilon}}$

<u>outside the sphere</u> $\vec{P} = 0$, $\epsilon = 1$ and

$$\vec{E} = \frac{Q}{r^2}\hat{r} \qquad r > R$$

There is thus a discontinuity in E at $r = R$.
This is due to the bound surface charge!

$$\vec{E}_{above} - \vec{E}_{below} = \left(\frac{Q}{R^2} - \frac{Q}{\epsilon R^2}\right)\hat{r} = 4\pi\sigma_b\,\hat{r}$$

$$\sigma_b = \frac{Q}{4\pi R^2}\left(1 - \frac{1}{\epsilon}\right) = \frac{Q}{4\pi R^2}\left(\frac{\epsilon - 1}{\epsilon}\right)$$

$$\sigma_b = \frac{Q}{4\pi R^2}\left(\frac{4\pi\chi_e}{\epsilon}\right) = \frac{Q\chi_e}{\epsilon R^2}$$

Let us check this from $\vec{\mathbb{P}}$ directly

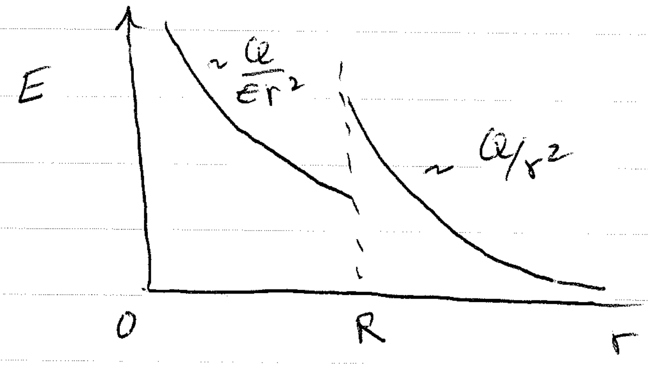
$$\sigma_b = \vec{\mathbb{P}}\cdot\hat{n} = (\chi_e\vec{E})\cdot\hat{r} = \frac{\chi_e Q}{\epsilon R^2}$$

agrees with above!
from discontinuity in $\vec{E}$

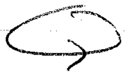
What is total bound charge at surface?

$$4\pi R^2\sigma_b = 4\pi R^2\,\frac{\chi_e Q}{\epsilon R^2} = \frac{4\pi\chi_e}{\epsilon}Q$$

This is just equal in magnitude but opposite in sign to the bound charge at the origin. This must be so since the total bound charge must always sum to zero.



Magnetic properties of materials

magnetic fields are created by circulating current loops. 

In matter, such circulating current loops can arise from two possible sources

1) orbital angular momentum of electrons about nucleus
2) intrinsic "spin" of electrons.

In most materials, these microscopic current loops are oriented so that the net magnetic field produced by them averages to zero on macroscopic length scales. But when one applies a magnetic field $\vec{B}$ there is a net orientation of loops to create a net magnetic field. One characterizes this by the "magnetization density" of the material $\vec{M}$.

magnetic moment of a current loop is

$$\vec{m} = \underset{\substack{\uparrow \\ \text{current circulating} \\ \text{loop}}}{I} \; \underset{\nwarrow \text{ area of loop}}{A} \; \hat{n} \leftarrow \text{normal to loop}$$

$$\vec{M} = \frac{\sum_i \vec{m}_i}{\Delta V} \quad \text{in small volume } \Delta V$$

when loops have random orientations, the average $\vec{M}$ sums to zero and the net magnetic field produced by all the current loops also vanishes.

But when the current loops orient to give a finite $\vec{M}$ then ~~there is fin~~ the magnetic fields from the loops will add to give a non-zero average magnetic field.

Unlike dielectrics, where the induced electric field is always opposite the applied ~~ma~~ electric field, ie $\chi_e > 0$, for magnetic materials the effect can go either way.

For $\vec{M} = \chi_m \vec{B}$ with $\chi_m > 0$ the material is called paramagnetic.
For $\chi_m < 0$ the material is called diamagnetic.

If $\vec{M} \neq 0$ even when $\vec{B} = 0$, the material is ferromagnetic. This is the case of bar magnets - the direction from South to North pole is the direction of net orientation of the microscopic current loops.